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Kinetic and magnetic α-effects in non-linear dynamo theory

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ABSTRACT

The backreaction of the Lorentz force on the α-effect is studied in the limit of small magnetic and fluid Reynolds numbers, using the first-order smoothing approximation (FOSA) to solve both the induction and momentum equations. Both steady and time-dependent forcings are considered. In the low Reynolds number limit, the velocity and magnetic fields can be expressed explicitly in terms of the forcing function. The non-linear α-effect is then shown to be expressible in several equivalent forms in agreement with formalisms that are used in various closure schemes. On one hand, one can express α completely in terms of the helical properties of the velocity field as in traditional FOSA, or, alternatively, as the sum of two terms, a so-called kinetic α-effect and an oppositely signed term proportional to the helical part of the small-scale magnetic field. These results hold for both steady and time-dependent forcing at arbitrary strength of the mean field. In addition, the τ-approximation is considered in the limit of small fluid and magnetic Reynolds numbers. In this limit, the τ closure term is absent and the viscous and resistive terms must be fully included. The underlying equations are then identical to those used under FOSA, but they reveal interesting differences between the steady and time-dependent forcing. For steady forcing, the correlation between the forcing function and the small-scale magnetic field turns out to contribute in a crucial manner to determine the net α-effect. However for delta-correlated time-dependent forcing, this force–field correlation vanishes, enabling one to write α exactly as the sum of kinetic and magnetic α-effects, similar to what one obtains also in the large Reynolds number regime in the τ-approximation closure hypothesis. In the limit of strong imposed fields, B_0, we find $\alpha \propto B_0^{-2}$ for delta-correlated forcing, in contrast to the well-known $\alpha \propto B_0^{-3}$ behaviour for the case of a steady forcing. The analysis presented here is also shown to be in agreement with numerical simulations of steady as well as random helical flows.

Key words: hydrodynamics – magnetic fields – MHD – turbulence.

1 INTRODUCTION

Turbulent mean field dynamos (MFDs) are thought to be at the heart of magnetic field generation and maintenance in most astrophysical bodies, like the sun or the galaxy. A particularly important driver of the MFD is the α-effect which, in the kinematic regime, depends only on the helical properties of the turbulence. It is crucial to understand how the α-effect gets modified due to the backreaction of the generated mean and fluctuating fields.

Using closure schemes or the quasi-linear approximation it has been argued that, due to Lorentz forces, the α-effect gets 'renormalized' by the addition of a term proportional to the current helicity of the generated small-scale magnetic fields (Pouquet, Frisch & Léorat 1976; Gruzinov & Diamond 1994; Kleeorin, Rogachevskii & Ruzmaikin 1995; Kleeorin & Rogachevskii 1999; Subramanian 1999; Blackman & Field 2000; Rädler, Kleeorin & Rogachevskii 2003; Brandenburg & Subramanian 2005a). The presence of such an additional term is uncontroversial if a helical small-scale magnetic field is present even in the absence of a mean field. However, it has been argued that, in the absence of such a pre-existing small-scale magnetic field, the α-effect can be expressed exclusively in terms of the velocity field, albeit one which is a solution of the full momentum equation including the Lorentz force (Proctor 2003; Rädler & Rheinhardt 2007). In the latter case, it is not obvious that the helicity of the small-scale magnetic field plays any explicit role in the backreaction to α. It is important to clarify this issue, as it will decide how one should understand the saturation of turbulent dynamos, as well as the possibility of catastrophic quenching of the α-effect and

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ways to alleviate such quenching. Here and below, 'catastrophic' means that α is quenched down to values on the order of the inverse magnetic Reynolds number.

In order to clarify these conflicting views, we examine here an exactly solvable model of the non-linear backreaction to the α-effect, where we assume small magnetic and fluid Reynolds numbers. Obviously, this approach does not allow us to address the question of catastrophic quenching of astrophysical dynamos directly, but it allows us to make novel and unambiguous statements that help clarifying the nature of magnetic saturation. We will show that, at least in this simple context, both the above viewpoints are consistent, if interpreted properly.

2 MEAN FIELD ELECTRODYNAMICS

In mean field electrodynamics (Moffatt 1978; Krause & Rädler 1980), any field $\boldsymbol{F}$ is split into a mean field $\overline{\boldsymbol{F}}$ and a 'fluctuating' small-scale field $\boldsymbol{f}$, such that $\boldsymbol{F} = \overline{\boldsymbol{F}} + \boldsymbol{f}$. The fluctuating velocity (or magnetic) field is assumed to possess a correlation length l small compared to the length scale L of the variation of the mean field. The magnetic field obeys the induction equation,

$$\frac{\partial \boldsymbol{B}}{\partial t} = \eta \nabla^2 \boldsymbol{B} + \nabla \times (\boldsymbol{U} \times \boldsymbol{B}), \quad \nabla \cdot \boldsymbol{B} = 0, \tag{1}$$

where $\boldsymbol{U}$ represents the fluid velocity, $\eta = (\mu_0 \sigma)^{-1}$ is the magnetic diffusivity (assumed constant), σ is the electric conductivity, and μ_0 is the vacuum permeability. Averaging equation (1), we obtain the standard MFD equation

$$\frac{\partial \overline{\boldsymbol{B}}}{\partial t} = \eta \nabla^2 \overline{\boldsymbol{B}} + \nabla \times (\overline{\boldsymbol{U}} \times \overline{\boldsymbol{B}} + \boldsymbol{\mathcal{E}}), \quad \nabla \cdot \overline{\boldsymbol{B}} = 0. \tag{2}$$

This averaged equation now has a new term, the mean electromotive force (emf) $\boldsymbol{\mathcal{E}} = \overline{\boldsymbol{u} \times \boldsymbol{b}}$, which crucially depends on the statistical properties of the $\boldsymbol{u}$ and $\boldsymbol{b}$ fields. The central closure problem in mean field theory is to find an expression for the correlator $\boldsymbol{\mathcal{E}}$ in terms of the mean fields.

To find an expression for $\boldsymbol{\mathcal{E}}$, one needs the evolution equations for both the fluctuating magnetic field $\boldsymbol{b}$ and the fluctuating velocity field $\boldsymbol{u}$. The first follows from subtracting equation (2) from equation (1),

$$\frac{\partial \boldsymbol{b}}{\partial t} = \eta \nabla^2 \boldsymbol{b} + \nabla \times (\overline{\boldsymbol{U}} \times \boldsymbol{b} + \boldsymbol{u} \times \overline{\boldsymbol{B}}) + \boldsymbol{G}, \quad \nabla \cdot \boldsymbol{b} = 0. \tag{3}$$

Here $\boldsymbol{G} = \nabla \times (\boldsymbol{u} \times \boldsymbol{b})'$ with $(\boldsymbol{u} \times \boldsymbol{b})' = \boldsymbol{u} \times \boldsymbol{b} - \overline{\boldsymbol{u} \times \boldsymbol{b}}$. In what follows, we will set the mean field velocity to zero, that is, $\overline{\boldsymbol{U}} = 0$ and focus solely on the effect of the fluctuating velocity.

The evolution equation for $\boldsymbol{u}$ can be derived in a similar manner by subtracting the averaged momentum equation from the full momentum equation. We assume the flow to be incompressible with $\nabla \cdot \boldsymbol{u} = 0$. We get

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\frac{1}{\rho} \nabla \left(p + \frac{1}{\mu_0} \overline{\boldsymbol{B}} \cdot \boldsymbol{b} \right) + \nu \nabla^2 \boldsymbol{u} + \frac{1}{\mu_0 \rho} \left[(\overline{\boldsymbol{B}} \cdot \nabla) \boldsymbol{b} + (\boldsymbol{b} \cdot \nabla) \overline{\boldsymbol{B}} \right] + \boldsymbol{f} + \boldsymbol{T}. \tag{4}$$

Here ρ is the mass density, p is the perturbed fluid pressure, ν is the kinematic viscosity taken to be constant, $\boldsymbol{f}$ is the fluctuating force, and

$$\boldsymbol{T} = -(\boldsymbol{u} \cdot \nabla \boldsymbol{u})' - \frac{1}{\mu_0 \rho} \left[(\boldsymbol{b} \cdot \nabla \boldsymbol{b})' - \tfrac{1}{2} \nabla (\boldsymbol{b}^2)' \right] \tag{5}$$

contains the second order terms in $\boldsymbol{u}$ and $\boldsymbol{b}$. Here, primed quantities indicate deviations from the mean, that is, $X' = X - \overline{X}$. We will also redefine $\boldsymbol{b}/\sqrt{\mu_0 \rho} \rightarrow \boldsymbol{b}$, by setting $\mu_0 \rho = 1$, so that the magnetic field is measured in velocity units.

In order to find $\boldsymbol{\mathcal{E}}$ under the influence of the Lorentz force one has to solve equations (3) and (4) simultaneously and compute $\overline{\boldsymbol{u} \times \boldsymbol{b}}$. In general this is a difficult problem and one has to take recourse to closure approximations or numerical simulations. To make progress we assume here $R_{\rm m} = ul/\eta \ll 1$ and $Re = ul/\nu \ll 1$; that is both the magnetic and fluid Reynolds numbers are small compared to unity. In this case there is no small-scale dynamo action and so the small-scale magnetic field is solely due to shredding the large-scale magnetic field. Here u and b (see below) are typical strengths of the fluctuating velocity and magnetic fields, respectively. In the low magnetic Reynolds number limit the ratio of the first non-linear term in $\boldsymbol{G}$ to the resistive term in equation (3) is $\sim (ub/l)/(\eta b/l^2) \sim R_{\rm m} \ll 1$. So this part of $\boldsymbol{G}$ can be neglected compared to the resistive term. (Note that the second term in $\boldsymbol{G}$ vanishes automatically when taking the averages to evaluate the mean emf.) Neglecting the non-linear term, the generation rate of b is $\sim u\overline{B}/l$, while its destruction rate is $\sim \eta b/l^2$. Equating these two rates, this also implies that $b \sim R_{\rm m} \overline{B}$ and the fluctuation field is only a small perturbation to mean fields. Similarly the ratio of the non-linear advection term to the viscous term in equation (4), is $\sim (u^2/l)/(\nu u/l^2) \sim Re \ll 1$ and the ratio of the parts of the Lorentz force non-linear in $\boldsymbol{b}$ to that linear in $\boldsymbol{b}$ is $\sim (b^2/l)/(b\overline{B}/l) \sim R_{\rm m} \ll 1$. So $\boldsymbol{T}$ can also be neglected in equation (4). In this limit, one can therefore apply the well-known first-order smoothing approximation (FOSA). It is sometimes also referred to as the second-order correlation approximation (or SOCA; see e.g. Krause & Rädler 1980). This approximation consists of neglecting the non-linear terms $\boldsymbol{G}$ and $\boldsymbol{T}$, to solve *both* the induction and momentum equation. Since FOSA is applied to the momentum equation as well, we will refer to this as 'double FOSA'. In order to make the problem analytically tractable, we will take $\overline{\boldsymbol{B}} = \boldsymbol{B}_0 =$ constant. This also allows us to isolate the α-effect in a straightforward fashion. In the next section we begin by considering for simplicity the case of steady forcing. It is then possible to also neglect the time derivatives in equations (3) and (4). We return to consider time-dependent forcing in detail in Section 4.

3 COMPUTING $\mathcal{E}$ FOR STEADY FORCING

Under the assumptions highlighted above, one can solve directly for $\boldsymbol{u}$ and $\boldsymbol{b}$ in terms of the forcing function $\boldsymbol{f}$. This in turn allows the calculation of the mean emf in four ways.

A. We use the induction equation to solve for $\boldsymbol{b}$ in terms of $\boldsymbol{u}$. Then one can write the emf completely in terms of the velocity field, as in normal FOSA and then substitute for $\boldsymbol{u}$ in terms of $\boldsymbol{f}$.

B. We compute $\boldsymbol{\mathcal{E}} = \overline{\boldsymbol{u} \times \boldsymbol{b}}$ directly.

C. We use the momentum equation to solve for $\boldsymbol{u}$ in terms of $\boldsymbol{b}$ and the forcing function $\boldsymbol{f}$, and then substitute for $\boldsymbol{b}$ in terms of $\boldsymbol{f}$.

D. Compute $\boldsymbol{\mathcal{E}}$ from the $\partial \boldsymbol{\mathcal{E}}/\partial t = 0$ relation, as in τ-approximation closures.

We will show that all four methods give the same answer for the mean emf in terms of the forcing function $\boldsymbol{f}$. Method A gives the traditional FOSA result for the α-effect being dependent on the helical properties of the velocity field, while Method C can be interpreted to reflect the idea of a renormalized α due to the helicity of small-scale magnetic fields. But we show that the final answer in terms of the forcing is identical.

Before going into the various methods as highlighted above, we solve for $\boldsymbol{u}$ and $\boldsymbol{b}$ in terms of the forcing function $\boldsymbol{f}$. In the low-conductivity limit, neglecting the time variation of $\boldsymbol{b}$ in equation (3) we have

$$-\eta\nabla^2\boldsymbol{b} = \boldsymbol{B}_0\cdot\nabla\boldsymbol{u}. \tag{6}$$

Similarly, in the limit of low Re and $R_{\rm m}$, equation (4) becomes

$$-\nu\nabla^2\boldsymbol{u} = \boldsymbol{B}_0\cdot\nabla\boldsymbol{b} + \boldsymbol{f} - \nabla p_{\rm eff}, \tag{7}$$

where $p_{\rm eff}$ combines the hydrodynamic and the magnetic pressure. Using the incompressibility condition, one can eliminate $p_{\rm eff}$. We will solve these equations in Fourier space. Throughout this paper we will be using the convention

$$\tilde{\boldsymbol{u}}(\boldsymbol{k}) = \frac{1}{(2\pi)^3}\int \boldsymbol{u}(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{d}\boldsymbol{x}, \tag{8}$$

which satisfies the inverse relation

$$\boldsymbol{u}(\boldsymbol{x}) = \int \tilde{\boldsymbol{u}}(\boldsymbol{k})\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{d}\boldsymbol{k}. \tag{9}$$

In Fourier space, equations (6) and (7) become

$$\eta k^2\tilde{b}_i(\boldsymbol{k}) = (\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{B}_0)\,\tilde{u}_i(\boldsymbol{k}), \tag{10}$$

$$\nu k^2\tilde{u}_i(\boldsymbol{k}) = (\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{B}_0)\,\tilde{b}_i(\boldsymbol{k}) + \tilde{f}_i(\boldsymbol{k}), \tag{11}$$

where we have chosen the forcing to be divergence free, with $\mathrm{i}\boldsymbol{k}\cdot\tilde{\boldsymbol{f}} = 0$. We can therefore solve the above two equations simultaneously to express $\tilde{\boldsymbol{u}}$ and $\tilde{\boldsymbol{b}}$ completely in terms of $\tilde{\boldsymbol{f}}$,

$$\tilde{u}_i(\boldsymbol{k}) = \frac{\tilde{f}_i(\boldsymbol{k})}{\nu k^2 + (\boldsymbol{B}_0\cdot\boldsymbol{k})^2/\eta k^2}, \tag{12}$$

$$\tilde{b}_i(\boldsymbol{k}) = \frac{\tilde{f}_i(\boldsymbol{k})}{\nu k^2 + (\boldsymbol{B}_0\cdot\boldsymbol{k})^2/\eta k^2}\,\frac{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{B}_0}{\eta k^2}. \tag{13}$$

We can use these solutions to calculate $\boldsymbol{\mathcal{E}}$. For getting an explicit expression, we also need the equal time force correlation function. For isotropic and homogeneous forcing, this is given by

$$\overline{\tilde{f}_j(\boldsymbol{p},t)\;\tilde{f}_k(\boldsymbol{q},t)} = \delta^3(\boldsymbol{p}+\boldsymbol{q})F_{jk}(\boldsymbol{q}). \tag{14}$$

Here, F_{jk} is the force spectrum tensor which is given by

$$F_{jk}(\boldsymbol{k}) = P_{jk}\frac{\Phi(k)}{4\pi k^2} + \epsilon_{jkm}\frac{\mathrm{i}k_m\chi(k)}{8\pi k^4}, \tag{15}$$

where $P_{jk} = \delta_{jk} - k_jk_k/k^2$ is the projection operator, and $\Phi(k)$ and $\chi(k)$ are spectra characterizing the mean squared value and the helicity of the forcing function, normalized such that

$$\int_0^\infty \Phi(k)\;\mathrm{d}k = \tfrac{1}{2}\overline{\boldsymbol{f}^2} \equiv \tfrac{1}{2}A_{\rm f}^2. \tag{16}$$

$$\int_0^\infty \chi(k)\;\mathrm{d}k = \overline{\boldsymbol{f}\cdot(\nabla\times\boldsymbol{f})} \equiv H_{\rm f}. \tag{17}$$

The mean emf can be written as

$$\mathcal{E}_i(\boldsymbol{x}) = \epsilon_{ijk}\overline{\boldsymbol{u}_j(\boldsymbol{x})\boldsymbol{b}_k(\boldsymbol{x})} = \int \tilde{\mathcal{E}}_i(\boldsymbol{k})\;\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{d}\,\boldsymbol{k}, \tag{18}$$

where the Fourier transform $\tilde{\boldsymbol{\mathcal{E}}}$ is given by

$$\tilde{\mathcal{E}}_i(\boldsymbol{k}) = \epsilon_{ijk}\int \overline{\tilde{u}_j(\boldsymbol{k}-\boldsymbol{q})\tilde{b}_k(\boldsymbol{q})}\;\mathrm{d}\boldsymbol{q}. \tag{19}$$

We now turn to the calculation of the non-linear mean emf and the resulting non-linear α-effect in the four different methods outlined above.

3.1 Method A: express $\boldsymbol{b}$ in terms of $\boldsymbol{u}$ and then solve for $\boldsymbol{\mathcal{E}}$

In this approach we use the induction equation to solve for $\boldsymbol{b}$ in terms of $\boldsymbol{u}$. Using equation (10) to express $\boldsymbol{b}$ in terms of $\boldsymbol{u}$ in equation (19) gives

$$\tilde{\mathcal{E}}_i(\boldsymbol{k}) = \mathrm{i}\epsilon_{ijk}\int \frac{\boldsymbol{B}_0\cdot\boldsymbol{q}}{\eta q^2}\;\overline{\tilde{u}_j(\boldsymbol{k}-\boldsymbol{q})\tilde{u}_k(\boldsymbol{q})}\;\mathrm{d}\boldsymbol{q}. \tag{20}$$

At this stage one can put the emf completely in terms of the velocity field and recover the usual FOSA expression that in the low-conductivity and isotropic limit, the α-effect is related to the helicity of the velocity potential (Krause & Rädler 1980; Rädler & Rheinhardt 2007). This can be shown in the following manner: since $\nabla\cdot\boldsymbol{u} = 0$, the velocity field can be expressed as $\boldsymbol{u} = \nabla\times\boldsymbol{\psi}$, where $\boldsymbol{\psi}$ is the velocity vector potential with the gauge condition $\nabla\cdot\boldsymbol{\psi} = 0$. We then have $\tilde{u}_k(\boldsymbol{q}) = \mathrm{i}q_p\epsilon_{kpl}\tilde{\psi}_l(\boldsymbol{q})$. Substituting this expression in equation (20), and using the fact that the velocity field is divergenceless, we get

$$\tilde{\mathcal{E}}_i(\boldsymbol{k}) = k_j\int \frac{\boldsymbol{B}_0\cdot\boldsymbol{q}}{\eta q^2}\;\overline{\tilde{u}_j(\boldsymbol{k}-\boldsymbol{q})\tilde{\psi}_i(\boldsymbol{q})}\;\mathrm{d}\boldsymbol{q} - \int \frac{\boldsymbol{B}_0\cdot\boldsymbol{q}}{\eta q^2}\;q_i\;\overline{\tilde{u}_l(\boldsymbol{k}-\boldsymbol{q})\tilde{\psi}_l(\boldsymbol{q})}\;\mathrm{d}\boldsymbol{q}. \tag{21}$$

For homogeneous and isotropic turbulence, the $\overline{\tilde{u}_j(\boldsymbol{k}-\boldsymbol{q})\tilde{\psi}_i(\boldsymbol{q})}$ correlation is proportional to $\delta^3(\boldsymbol{k})$. Since the first term in equation (21) is $\propto k_j$, it does not contribute to $\boldsymbol{\mathcal{E}}$. Therefore,

$$\tilde{\mathcal{E}}_i(\boldsymbol{k}) = -\int \frac{B_{0m}\;q_m}{\eta q^2}\;q_i\;\overline{\tilde{u}_l(\boldsymbol{k}-\boldsymbol{q})\tilde{\psi}_l(\boldsymbol{q})}\;\mathrm{d}\boldsymbol{q}. \tag{22}$$

Again, for homogeneous and isotropic turbulence, the $\overline{\tilde{u}_l(\boldsymbol{k}-\boldsymbol{q})\tilde{\psi}_l(\boldsymbol{q})}$ correlation is $\propto\delta^3(\boldsymbol{k})g(|\boldsymbol{q}|)$. One can then carry out the angular integral in equation (22) using $\int(q_mq_i/q^2)(\mathrm{d}\Omega/4\pi) = (1/3)\delta_{mi}$ to get $\mathcal{E}_i(\boldsymbol{x}) = \alpha B_{0i}$, where α is given by

$$\alpha = -\frac{1}{3\eta}\;\overline{\boldsymbol{\psi}\cdot\boldsymbol{u}}, \tag{23}$$

which is identical to the expressions obtained by Krause & Rädler (1980); see also Rädler & Rheinhardt (2007). (Note that when the Lorentz force becomes important the assumption of isotropy in the above derivation breaks down and the α-effect becomes anisotropic, as calculated below and detailed in Section 3.5.)

Since we have already solved for the velocity field explicitly, we can now derive an expression for the mean emf in this approach. Substituting the velocity in terms of the forcing function, the mean emf in coordinate space is given by

$$\mathcal{E}_i(\boldsymbol{x}) = \mathrm{i}\epsilon_{ijk}\int \frac{\boldsymbol{B}_0\cdot\boldsymbol{q}}{\eta q^2}\frac{F_{jk}(\boldsymbol{q})}{[\nu q^2 + (\boldsymbol{B}_0\cdot\boldsymbol{q})^2/\eta q^2]^2}\;\mathrm{d}\boldsymbol{q}. \tag{24}$$

Here F_{jk} the spectrum tensor for the force–field given by equation (15). Note that only the antisymmetric part of F_{jk} contributes to $\boldsymbol{\mathcal{E}}$, due to the presence of ϵ_{ijk} on the right-hand side (RHS) of the above equation. We can also write $\boldsymbol{\mathcal{E}}$ as

$$\mathcal{E}_i(\boldsymbol{x}) = \mathrm{i}\epsilon_{ijk}\int \frac{\boldsymbol{B}_0\cdot\boldsymbol{q}}{(\eta q^2)(\nu q^2)^2}\frac{F_{jk}(\boldsymbol{q})}{[1+N]^2}\;\mathrm{d}\boldsymbol{q}, \tag{25}$$

where $N = (\boldsymbol{B}_0\cdot\boldsymbol{q})^2/(\eta\nu q^4)$ determines the importance of the Lorentz forces on the mean emf. It is to be noted that the limit of small Lorentz forces corresponds to taking $N \ll 1$ above.

3.2 Method B: compute $\mathcal{E}$ directly

In this approach, we directly compute $\boldsymbol{\mathcal{E}} = \overline{\boldsymbol{u} \times \boldsymbol{b}}$ by substituting $\boldsymbol{u}$ and $\boldsymbol{b}$ in terms of $\boldsymbol{f}$, using equations (12) and (13). We then get

$$\tilde{\mathcal{E}}_i(\boldsymbol{k}) = \mathrm{i}\epsilon_{ijk} \int \frac{\boldsymbol{B}_0 \cdot \boldsymbol{q}}{\eta q^2} \frac{\overline{\tilde{f}_j(\boldsymbol{k}-\boldsymbol{q})\tilde{f}_k(\boldsymbol{q})}}{\gamma(\boldsymbol{k}-\boldsymbol{q})\gamma(\boldsymbol{q})}\, \mathrm{d}\boldsymbol{q}, \tag{26}$$

where we have defined $\gamma(\boldsymbol{q}) = \nu q^2 + (\boldsymbol{B}_0 \cdot \boldsymbol{q})^2/\eta q^2$. Substituting for the force correlation, the mean emf in coordinate space is given by

$$\mathcal{E}_i(\boldsymbol{x}) = \mathrm{i}\epsilon_{ijk} \int \frac{\boldsymbol{B}_0 \cdot \boldsymbol{q}}{\eta q^2} \frac{F_{jk}(\boldsymbol{q})}{[\nu q^2 + (\boldsymbol{B}_0 \cdot \boldsymbol{q})^2/\eta q^2]^2}\, \mathrm{d}\boldsymbol{q}, \tag{27}$$

which is identical to equation (24) for $\boldsymbol{\mathcal{E}}$ obtained by Method A.

3.3 Method C: express $\boldsymbol{u}$ in terms of $\boldsymbol{b}$ and then solve for $\mathcal{E}$

Note that one could also start from the momentum equation to compute $\boldsymbol{\mathcal{E}}$. In this approach, we first solve for $\boldsymbol{u}$ in terms of $\boldsymbol{b}$ and the forcing function $\boldsymbol{f}$, and then substitute for $\boldsymbol{b}$ in terms of $\boldsymbol{f}$ using equation (13). The difference from the earlier treatments will be an additional term containing an $\boldsymbol{f} \times \boldsymbol{b}$-like correlation, which turns out to be essential for calculating the $\boldsymbol{\mathcal{E}}$ correctly. Using equation (11) one can write

$$\tilde{u}_i(\boldsymbol{k}) = \frac{1}{\nu k^2}[(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{B}_0)\,\tilde{b}_i(\boldsymbol{k}) + \tilde{f}_i(\boldsymbol{k})]. \tag{28}$$

From equation (19) the mean emf can then be written as

$$\begin{aligned}\tilde{\mathcal{E}}_i(\boldsymbol{k}) = {}& \epsilon_{ijk} \int \frac{1}{\nu(\boldsymbol{k}-\boldsymbol{q})^2} \overline{\tilde{f}_j(\boldsymbol{k}-\boldsymbol{q})\tilde{b}_k(\boldsymbol{q})}\, \mathrm{d}\boldsymbol{q} \\ & + \mathrm{i}\epsilon_{ijk} \int \frac{\boldsymbol{B}_0 \cdot (\boldsymbol{k}-\boldsymbol{q})}{\nu(\boldsymbol{k}-\boldsymbol{q})^2} \overline{\tilde{b}_j(\boldsymbol{k}-\boldsymbol{q})\tilde{b}_k(\boldsymbol{q})}\, \mathrm{d}\boldsymbol{q}.\end{aligned} \tag{29}$$

Here the first term involves the $\boldsymbol{f} \times \boldsymbol{b}$-like correlation. To elucidate the meaning of the second term it is useful to define the magnetic field as $\boldsymbol{b} = \nabla \times \boldsymbol{a}$, where $\boldsymbol{a}$ is the small-scale magnetic vector potential in the Coulomb gauge ($\nabla \cdot \boldsymbol{a} = 0$). Then, for isotropic small-scale fields, following the approach in Method A, the second term in equation (29) gives a contribution to $\boldsymbol{\mathcal{E}}$ of the form $\hat{\alpha}_{\mathrm{M}}\boldsymbol{B}_0$, where

$$\hat{\alpha}_{\mathrm{M}} = \frac{1}{3\nu}\,\overline{\boldsymbol{a} \cdot \boldsymbol{b}}. \tag{30}$$

So this contribution is proportional to the magnetic helicity of the small-scale magnetic field (analogous to the helicity of the vector potential of the velocity field).

If we substitute $\boldsymbol{b}$ in terms of $\boldsymbol{f}$ from equation (13) and then integrate over the delta function, the mean emf in coordinate space can be expressed as

$$\begin{aligned}\mathcal{E}_i(\boldsymbol{x}) = {}& \mathrm{i}\epsilon_{ijk} \int \frac{\boldsymbol{B}_0 \cdot \boldsymbol{q}}{\eta q^2} \frac{F_{jk}(\boldsymbol{q})}{(\nu q^2)^2\ [1+N]} \mathrm{d}\boldsymbol{q} \\ & - \mathrm{i}\epsilon_{ijk} \int \frac{\boldsymbol{B}_0 \cdot \boldsymbol{q}}{\eta q^2} \frac{F_{jk}(\boldsymbol{q})}{(\nu q^2)^2\ [1+N]} \frac{N}{1+N}\, \mathrm{d}\boldsymbol{q}.\end{aligned} \tag{31}$$

The two terms on the RHS of the above equation have an interesting interpretation. As mentioned above, the limit of small Lorentz forces corresponds to taking $N = (\boldsymbol{B}_0 \cdot \boldsymbol{q})^2/\eta\nu q^4 \ll 1$. In this limit the second integral vanishes while the first one [i.e. the $\boldsymbol{f} \times \boldsymbol{b}$-like correlation in equation (31), which is really a $(\nabla^{-2}\boldsymbol{f}) \times \boldsymbol{b}$ correlation] goes over to a kinematic α-effect. [One can see by comparing equation (25) and the first term in equation (31) that the two are identical in the limit $N \ll 1$.] In fact, this part of the α-effect can be obtained from equations (12) and (13) by neglecting the Lorentz force in the expression for $\tilde{u}_i(\boldsymbol{k})$. In the following we refer to the contribution of the field-aligned component of this term divided by B_0, as the $\hat{\alpha}_{\mathrm{F}}$ term, because it comes from the $\boldsymbol{f} \times \boldsymbol{b}$-like correlation.

As N is increased the contribution from the first term decreases. In the same limit, the second term, which depends on the magnetic helicity, gains in importance. Since it has the opposite sign, it partially cancels the first term and further suppresses the total α-effect. This is reminiscent of the suppression of the kinetic alpha due to the addition of a magnetic alpha (proportional to helical part of $\boldsymbol{b}$) found in several closure models (Pouquet et al. 1976; Kleeorin & Ruzmaikin 1982; Gruzinov & Diamond 1994; Blackman & Field 2002; Brandenburg & Subramanian 2005b).

When adding the two terms in equation (31), the mean emf turns out to be

$$\mathcal{E}_i(\boldsymbol{x}) = \mathrm{i}\epsilon_{ijk} \int \frac{\boldsymbol{B}_0 \cdot \boldsymbol{q}}{\eta q^2} \frac{F_{jk}(q)}{[\nu q^2 + (\boldsymbol{B}_0 \cdot \boldsymbol{q})^2/\eta q^2]^2}\, \mathrm{d}\boldsymbol{q}, \tag{32}$$

which is identical to the expressions obtained in Methods A and B; see equations (24) and (27), respectively.

3.4 Method D: compute $\mathcal{E}$ from the $\partial\mathcal{E}/\partial t = 0$ relation

In recent years the so-called τ-approximation has received increased attention (Kleeorin, Mond & Rogachevskii 1996; Blackman & Field 2002; Rädler, Kleeorin & Rogachevskii 2003; Brandenburg & Subramanian 2005a; Rädler & Rheinhardt 2007). This involves invoking a closure whereby triple correlations which arise during the evaluation of $\partial\boldsymbol{\mathcal{E}}/\partial t$, are assumed to provide a damping term proportional to $\boldsymbol{\mathcal{E}}$ itself. In the present context there is no need to invoke a closure for the triple correlations, because these terms are small for low fluid and magnetic Reynolds numbers. It turns out that the correct expression for $\boldsymbol{\mathcal{E}}$ can still be derived in the same framework, where one evaluates the $\partial\boldsymbol{\mathcal{E}}/\partial t$ expression.

The expression for $\partial\boldsymbol{\mathcal{E}}/\partial t$ is governed by two terms, $\overline{\dot{\boldsymbol{u}} \times \boldsymbol{b}}$ and $\overline{\boldsymbol{u} \times \dot{\boldsymbol{b}}}$, where dots denote partial time differentiation. Of course, both $\dot{\boldsymbol{u}}$ an $\dot{\boldsymbol{b}}$ vanish in the present case, but this is the result of a cancellation of driving and dissipating terms. In the present analysis both terms will be retained, because the dissipating term, which is related to the desired $\boldsymbol{\mathcal{E}}$, can then just be written as the negative of the driving term.

We perform the analysis in Fourier space and begin by defining $\boldsymbol{E}(\boldsymbol{k}, \boldsymbol{q}) = \overline{\tilde{\boldsymbol{u}}(\boldsymbol{k}-\boldsymbol{q}) \times \tilde{\boldsymbol{b}}(\boldsymbol{q})}$. Note the required

$$\boldsymbol{\mathcal{E}} = \int \boldsymbol{E}(\boldsymbol{k}, \boldsymbol{q}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \mathrm{d}\boldsymbol{k}\, \mathrm{d}\boldsymbol{q}. \tag{33}$$

To calculate the time derivative $\partial\boldsymbol{\mathcal{E}}/\partial t$, one needs to evaluate $\dot{\boldsymbol{E}} = \dot{\boldsymbol{E}}_{\mathrm{K}} + \dot{\boldsymbol{E}}_{\mathrm{M}}$, where

$$\dot{\boldsymbol{E}}_{\mathrm{K}}(\boldsymbol{k}, \boldsymbol{q}) = \overline{\tilde{\boldsymbol{u}}(\boldsymbol{k}-\boldsymbol{q}) \times \dot{\tilde{\boldsymbol{b}}}(\boldsymbol{q})}, \tag{34}$$

$$\dot{\boldsymbol{E}}_{\mathrm{M}}(\boldsymbol{k}, \boldsymbol{q}) = \overline{\dot{\tilde{\boldsymbol{u}}}(\boldsymbol{k}-\boldsymbol{q}) \times \tilde{\boldsymbol{b}}(\boldsymbol{q})}. \tag{35}$$

For $\dot{\tilde{\boldsymbol{u}}}$ and $\dot{\tilde{\boldsymbol{b}}}$ we restore the time derivatives in equations (10) and (11), and obtain

$$\dot{\boldsymbol{E}}_{\mathrm{K}} = \mathrm{i}\boldsymbol{q} \cdot \boldsymbol{B}_0\, [\overline{\tilde{\boldsymbol{u}}(\boldsymbol{k}-\boldsymbol{q}) \times \tilde{\boldsymbol{u}}(\boldsymbol{q})}] - \eta q^2 \boldsymbol{E}, \tag{36}$$

$$\begin{aligned}\dot{\boldsymbol{E}}_{\mathrm{M}} = {}& \mathrm{i}(\boldsymbol{k}-\boldsymbol{q}) \cdot \boldsymbol{B}_0\, [\overline{\tilde{\boldsymbol{b}}(\boldsymbol{k}-\boldsymbol{q}) \times \tilde{\boldsymbol{b}}(\boldsymbol{q})}] - \nu(\boldsymbol{k}-\boldsymbol{q})^2 \boldsymbol{E} \\ & + \overline{\tilde{\boldsymbol{f}}(\boldsymbol{k}-\boldsymbol{q}) \times \tilde{\boldsymbol{b}}(\boldsymbol{q})}.\end{aligned} \tag{37}$$

Since all time derivatives are negligible, we can simplify the RHS of the above equations by using equations (12) and (13) to express

$\tilde{\boldsymbol{u}}$ and $\tilde{\boldsymbol{b}}$ in terms of the forcing function. Adding the two equations (36) and (37) yields

$$[\eta q^2 + \nu(\boldsymbol{k}-\boldsymbol{q})^2]\boldsymbol{E}_i = \delta^3(\boldsymbol{k})\frac{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{B}_0}{\eta q^2}\frac{\epsilon_{ijk}\,F_{jk}(\boldsymbol{q})}{\gamma(\boldsymbol{q})\gamma(\boldsymbol{k}-\boldsymbol{q})} \times \left[\eta q^2 + \gamma(\boldsymbol{k}-\boldsymbol{q}) - \frac{[(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{B}_0]^2}{\eta(\boldsymbol{k}-\boldsymbol{q})^2}\right], \tag{38}$$

where the function γ was defined just below equation (26). The expression in the squared brackets on the right hand side of the expression above exactly reduces to $\eta q^2 + \nu\,(\boldsymbol{k}-\boldsymbol{q})^2$ and so, in the steady state limit, $\partial/\partial t = 0$, we have

$$\boldsymbol{E}_i(\boldsymbol{k},\boldsymbol{q}) = \delta^3(\boldsymbol{k})\frac{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{B}_0}{\eta q^2}\frac{\epsilon_{ijk}\,F_{jk}(\boldsymbol{q})}{\gamma(\boldsymbol{q})\gamma(\boldsymbol{k}-\boldsymbol{q})}. \tag{39}$$

Using this expression in equation (33), and integrating over $\boldsymbol{k}$, we again recover the form of $\boldsymbol{\mathcal{E}}$ identical to Methods A, B and C.

Thus, in this simple example where one can apply FOSA to both the induction and momentum equations, one gets identical expression for $\boldsymbol{\mathcal{E}}$ in terms of the correlation properties of the forcing function $\boldsymbol{f}$, in all the four methods.

3.5 The non-linear α-effect

We now compute the non-linear α-effect explicitly from the expression of $\boldsymbol{\mathcal{E}}$ as obtained in the four methods discussed above. As has been mentioned earlier, only the antisymmetric part of F_{jk} contributes to $\boldsymbol{\mathcal{E}}$ in equation (15), so equation (32) takes the simple form

$$\mathcal{E}_i(\boldsymbol{x}) = \mathrm{i}\epsilon_{ijk}\int \frac{\boldsymbol{B}_0\cdot\boldsymbol{q}}{\eta q^2}\frac{\mathrm{i}q_m\epsilon_{kjm}\chi(q)}{8\pi q^4\left[\nu q^2 + (\boldsymbol{B}_0\cdot\boldsymbol{q})^2/\eta q^2\right]^2}\,\mathrm{d}\boldsymbol{q}. \tag{40}$$

Contraction between the two ϵ terms and solving for $\alpha = \boldsymbol{\mathcal{E}}\cdot\boldsymbol{B}_0/B_0^2$ leads to

$$\alpha = -\int \frac{\chi(q)}{\eta\nu^2 q^6}\frac{(\hat{\boldsymbol{B}}_0\cdot\hat{\boldsymbol{q}})^2}{\left[1+(\hat{\boldsymbol{B}}_0\cdot\hat{\boldsymbol{q}})^2\beta^2\right]^2}\frac{\mathrm{d}\boldsymbol{q}}{4\pi q^2}, \tag{41}$$

where we have introduced $\beta^2 = B_0^2/(\eta\nu q^2)$ and hats denote unit vectors. The solution involves an angular integral with respect to the cosine of the polar angle, $\mu = \hat{\boldsymbol{B}}_0\cdot\hat{\boldsymbol{q}}$,

$$F(\beta) = \int_{-1}^{1}\frac{\mu^2\,\mathrm{d}\mu}{(1+\beta^2\mu^2)^2} = \frac{1}{\beta^2}\left(\frac{\tan^{-1}\beta}{\beta} - \frac{1}{1+\beta^2}\right), \tag{42}$$

so that

$$\alpha = -\frac{1}{2\eta\nu^2}\int_0^\infty \frac{\chi(q)}{q^6}\,F(\beta)\,\mathrm{d}q. \tag{43}$$

Note that for small values of β we have $F(\beta) \approx 2/3 - 4\beta^2/5$. In the limit of large values of B_0 and β we have $F(\beta) \to \pi/(2\beta^3)$, so the expression of α reduces to

$$\alpha \to -\frac{\pi}{4B_0^3}\sqrt{\frac{\eta}{\nu}}\int_0^\infty \frac{\chi(q)}{q^3}\,\mathrm{d}q \quad (for\, B_0 \to \infty). \tag{44}$$

So, in the asymptotic limit of large B_0, we have $\alpha \to B_0^{-3}$. This is a well-known result that goes back to the pioneering works of Moffatt (1972) and Rüdiger (1974); see also Rüdiger & Kitchatinov (1993). Note that $\boldsymbol{\mathcal{E}} \times \boldsymbol{B}_0 = \boldsymbol{0}$, because the corresponding angular integral would be over the product of a sine and cosine term which vanishes.

To illustrate further the dependence of α on B_0 we need to adopt some form for the spectrum $\chi(q)$. We assume that the forcing is at a particular wavenumber, q_0, and choose $\chi(q) = H_\mathrm{f}\delta(q-q_0)$ where H_f is the helicity of the forcing. Then the integration of the delta function simply gives

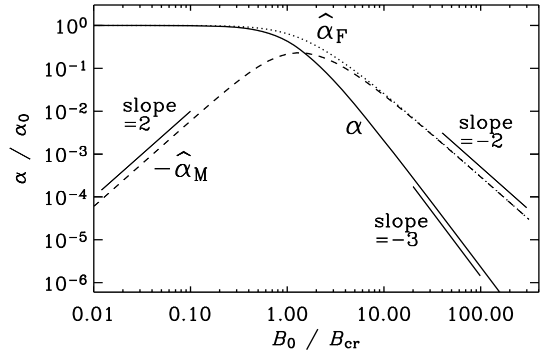

Figure 1. Variation of α, $\hat{\alpha}_\mathrm{F}$ and $-\hat{\alpha}_\mathrm{M}$ with B_0 from the analytical theory. Note the mutual approach of $\hat{\alpha}_\mathrm{F}$ and $-\hat{\alpha}_\mathrm{M}$ (asymptotic slope -2) to produce a lower, quenched value (asymptotic slope -3).

$$\frac{\alpha}{\alpha_0} = \frac{3}{2}\left(\frac{B_0}{B_\mathrm{cr}}\right)^{-2}\left[\frac{\tan^{-1}(B_0/B_\mathrm{cr})}{B_0/B_\mathrm{cr}} - \frac{1}{1+B_0^2/B_\mathrm{cr}^2}\right], \tag{45}$$

where we have defined

$$\alpha_0 = -H_\mathrm{f}/\left(3\eta\nu^2 q_0^6\right), \quad B_\mathrm{cr} = \sqrt{\eta\nu}\,q_0. \tag{46}$$

If we express $\alpha = \hat{\alpha}_\mathrm{F} + \hat{\alpha}_\mathrm{M}$, where $\hat{\alpha}_\mathrm{F}$ is computed from the $(\nabla^{-2}\,\boldsymbol{f})\times\boldsymbol{b}$ term and $\hat{\alpha}_\mathrm{M}$ from the $(\nabla^{-2}\,\boldsymbol{b})\times\boldsymbol{b}$ term in equation (31), we have

$$\frac{\hat{\alpha}_\mathrm{F}}{\alpha_0} = 3\left(\frac{B_0}{B_\mathrm{cr}}\right)^{-2}\left[1 - \frac{\tan^{-1}(B_0/B_\mathrm{cr})}{B_0/B_\mathrm{cr}}\right], \tag{47}$$

$$\frac{\hat{\alpha}_\mathrm{M}}{\alpha_0} = 3\left(\frac{B_0}{B_\mathrm{cr}}\right)^{-2}\left[\frac{3}{2}\frac{\tan^{-1}(B_0/B_\mathrm{cr})}{B_0/B_\mathrm{cr}} - \frac{3/2 + B_0^2/B_\mathrm{cr}^2}{1+B_0^2/B_\mathrm{cr}^2}\right]. \tag{48}$$

The hats on the αs indicate that a special choice has been made to divide α up into different contributions. A different choice without hats that had been derived under the τ approximation, will be discussed in Section 4.5.

We plot in Fig. 1 the variation of α with B_0. This shows that $\alpha \sim \alpha_0$ for $B_0 \lesssim B_\mathrm{cr}$ and in the asymptotic limit α decreases $\propto B_0^{-3}$. This figure also shows the variations of $\hat{\alpha}_\mathrm{F}$ and $\hat{\alpha}_\mathrm{M}$ with B_0 as predicted from equations (47) and (48). Both decrease asymptotically like B_0^{-2} because here, unlike in equation (45), the term in squared brackets remains constant. Their sum decreases as B_0^{-3}. One could also define a kinetic $\hat{\alpha}_\mathrm{K}$ from the $(\nabla^{-2}\,\boldsymbol{u})\times\boldsymbol{u}$ term in equation (25). In this case, for steady forcing we have $\alpha = \hat{\alpha}_\mathrm{K} = \hat{\alpha}_\mathrm{F} + \hat{\alpha}_\mathrm{M}$.

3.6 Comparison with simulations

Simulations allow us to alleviate some of the restrictions imposed by the analytical approach such as the limit of low fluid and magnetic Reynolds numbers, but they also introduce additional restrictions related for example to the degree of anisotropy. We adopt here a simple and frequently used steady and monochromatic forcing function that is related to an ABC flow, that is,

$$\boldsymbol{f}(\boldsymbol{x}) = \frac{A_\mathrm{f}}{\sqrt{3}}\begin{pmatrix}\sin k_\mathrm{f}z + \cos k_\mathrm{f}y\\ \sin k_\mathrm{f}x + \cos k_\mathrm{f}z\\ \sin k_\mathrm{f}y + \cos k_\mathrm{f}x\end{pmatrix}, \tag{49}$$

where A_f denotes the amplitude and k_f the wavenumber of the forcing function. This forcing function is isotropic with respect to the

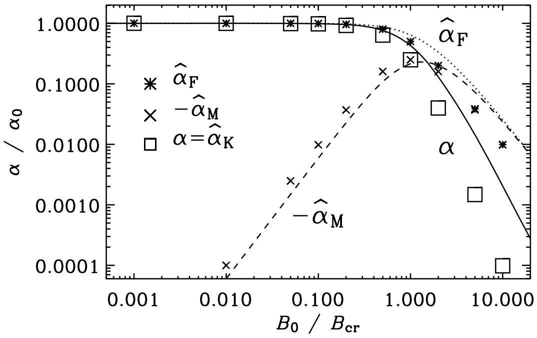

Figure 2. Variation of α, $\hat{\alpha}_{\rm F}$ and $-\hat{\alpha}_{\rm M}$ with B_0 for the ABC flow at low fluid and magnetic Reynolds numbers ($Re = R_{\rm m} = 10^{-4}$), compared with the analytic theory predicted for a fully isotropic flow. Note that the numerically determined values of $\hat{\alpha}_{\rm F}$ are smaller and those of $-\hat{\alpha}_{\rm M}$ larger than the corresponding analytic values. However, they still add up to satisfy the relation $\hat{\alpha}_{\rm F} + \hat{\alpha}_{\rm M} = \alpha$, predicted by analytic theory. In all cases the numerically determined values of $\hat{\alpha}_{\rm K}$ agree with the numerically determined values of α.

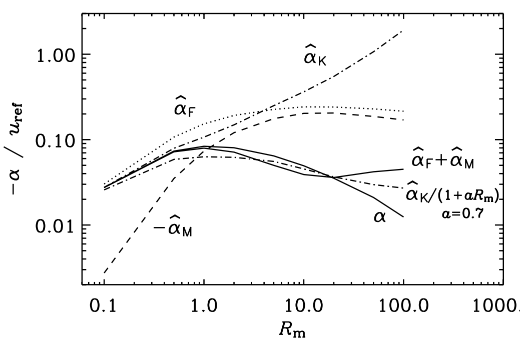

Figure 3. Dependence of the α-effect on $R_{\rm m}$ for fixed field strength, $B_0 = u_{\rm ref}$, for ABC-flow forcing at $Re = 1$. Note the agreement between α and $\hat{\alpha}_{\rm F} + \hat{\alpha}_{\rm M}$ as well as $\hat{\alpha}_{\rm K}/(1 + aR_{\rm m})$ for $R_{\rm m} \leqslant 30$.

three coordinate directions, but not with respect to other directions. The helicity of this forcing function is $H_{\rm f} = k_{\rm f} A_{\rm f}^2$. We use the PENCIL CODE,[1] which is a high-order finite-difference code (sixth order in space and third order in time) for solving the compressible hydromagnetic equations. We adopt a box size of $(2\pi)^3$ and take $A_{\rm f} = 10^{-4}$, $k_{\rm f} = 1$, and determine $\alpha = \mathcal{E}_z/B_{0z}$ as well as $\hat{\alpha}_{\rm K}$, $\hat{\alpha}_{\rm F}$ and $\hat{\alpha}_{\rm M}$, which are given, respectively, by the three integrals in equations (20) and (29). The result is shown in Fig. 2. For these runs a resolution of just 32^3 mesh points is sufficient, as demonstrated by comparing with runs with 64^3 mesh points.

For $B_0/B_{\rm cr} \leqslant 1$ the resulting values of α agree in all cases perfectly with both $\hat{\alpha}_{\rm K}$ and $\hat{\alpha}_{\rm F} + \hat{\alpha}_{\rm M}$. However, for $B_0/B_{\rm cr} > 1$ the numerically determined values of α are smaller than those expected theoretically using $q_0 = k_{\rm f}$. As in the analytical theory, the quenching is explained by an uprise of $-\hat{\alpha}_{\rm M}$. Note, however, that in the simulations this quantity attains a maximum at somewhat weaker field strength than in the analytical theory; cf. the dashed line and crosses in Fig. 2. We believe that this discrepancy is explained by an insufficient degree of isotropy of the forcing function.

Finally, it is interesting to address the question of the Reynolds number dependence of the quenching behaviour. In Fig. 3 we show the results for α, $\hat{\alpha}_{\rm K}$, $\hat{\alpha}_{\rm F}$ and $-\hat{\alpha}_{\rm M}$. Since the velocity is of the order of $u_{\rm ref} \equiv A_{\rm f}/(\nu k_{\rm f}^2)$, we have defined the fluid and magnetic Reynolds numbers as $Re = u_{\rm ref}/(\nu k_{\rm f})$ and $R_{\rm m} = u_{\rm ref}/(\eta k_{\rm f})$, respectively. For all runs we have assumed $Re = 1$ and $B_0 = u_{\rm ref}$.

Again, we see quite clearly the approach of $-\hat{\alpha}_{\rm M}$ towards $\hat{\alpha}_{\rm F}$, so as to make their sum diminish towards α with increasing values of $R_{\rm m}$. For $R_{\rm m} < 1$ the numerical data agree well with the analytic ones, whilst for $R_{\rm m} > 1$ the numerical values for all alphas lie below the analytic ones (not shown here). In particular, for $R_{\rm m} > 1$ the value of $\hat{\alpha}_{\rm K}$, based on the integral in equation (20), begins to exceed the value of α. This apparently signifies the breakdown of FOSA. However, one may expect that the relevant inverse time-scales or rates are no longer governed by just the resistive rate, $\sim\eta k_{\rm f}^2$, but also by a dynamical rate, $\sim u_{\rm ref}\, k_{\rm f}$. This leads to a correction factor, $1/(1 + aR_{\rm m})$, where $a \approx 1$ is an empirically determined coefficient quantifying the importance of this correction. In Fig. 3 we show that both $\hat{\alpha}_{\rm F} + \hat{\alpha}_{\rm M}$ as well as $\hat{\alpha}_{\rm K}/(1 + aR_{\rm m})$ with $a = 0.7$ are close to α for $R_{\rm m} \leqslant 30$. Note that no correction is necessary for $\hat{\alpha}_{\rm F}$ or $\hat{\alpha}_{\rm M}$, because these quantities are determined by the momentum equation and hence the viscous time-scale. However, since Re is small, no correction is necessary here. Again, a numerical resolution of 32^3 mesh points was used except for $R_{\rm m} \geqslant 10$, where we used 64^3 mesh points.

4 TIME-DEPENDENT FORCING

We now consider the case when $\boldsymbol{f}$ depends on time, but is nevertheless statistically stationary. In that case, both $\dot{\boldsymbol{b}}$ and $\dot{\boldsymbol{u}}$ are finite, and hence also $\overline{\boldsymbol{u} \times \dot{\boldsymbol{b}}}$ and $\overline{\dot{\boldsymbol{u}} \times \boldsymbol{b}}$ can in general be finite, even though their sum might vanish in the statistically steady state. Later we specialize to one case of particular interest, when the correlation time of the forcing is small. This was the case, for example, in the simulations of Brandenburg (2001) and Brandenburg & Subramanian (2005b). As in Section 3, we will assume here small magnetic and fluid Reynolds numbers and neglect the non-linear terms in the induction and momentum equations, but retain the time dependence. We will also now take a Fourier transform in time and define

$$\tilde{\boldsymbol{u}}(\boldsymbol{k}, \omega) = \frac{1}{(2\pi)^4} \int \boldsymbol{u}(\boldsymbol{x}, t) {\rm e}^{-{\rm i}\boldsymbol{k}\cdot\boldsymbol{x} + {\rm i}\omega t} {\rm d}\boldsymbol{x}\, {\rm d}t, \tag{50}$$

which satisfies the inverse relation

$$\boldsymbol{u}(\boldsymbol{x}, t) = \int \tilde{\boldsymbol{u}}(\boldsymbol{k}, \omega) {\rm e}^{{\rm i}\boldsymbol{k}\cdot\boldsymbol{x} - {\rm i}\omega t} {\rm d}\boldsymbol{k}\, {\rm d}\omega. \tag{51}$$

In Fourier space, equations (10) and (11) become

$$(-{\rm i}\omega + \eta k^2)\tilde{b}_i(\boldsymbol{k}, \omega) = ({\rm i}\boldsymbol{k} \cdot \boldsymbol{B}_0)\, \tilde{u}_i(\boldsymbol{k}, \omega), \tag{52}$$

$$(-{\rm i}\omega + \nu k^2)\tilde{u}_i(\boldsymbol{k}, \omega) = ({\rm i}\boldsymbol{k} \cdot \boldsymbol{B}_0)\, \tilde{b}_i(\boldsymbol{k}, \omega) + \tilde{f}_i(\boldsymbol{k}, \omega). \tag{53}$$

In order to simplify the writing of the equations below, it is convenient to define complex frequencies

$$\Gamma_\eta = -{\rm i}\omega + \eta k^2, \quad \Gamma_\nu = -{\rm i}\omega + \nu k^2. \tag{54}$$

As before, we can solve equations (52) and (53) simultaneously to express $\tilde{\boldsymbol{u}}$ and $\tilde{\boldsymbol{b}}$ completely in terms of $\tilde{\boldsymbol{f}}$,

$$\tilde{u}_i(\boldsymbol{k}, \omega) = \frac{\tilde{f}_i(\boldsymbol{k}, \omega)}{\Gamma_\nu + (\boldsymbol{B}_0 \cdot \boldsymbol{k})^2/\Gamma_\eta}, \tag{55}$$

$$\tilde{b}_i(\boldsymbol{k}, \omega) = \frac{\tilde{f}_i(\boldsymbol{k}, \omega)}{\Gamma_\nu + (\boldsymbol{B}_0 \cdot \boldsymbol{k})^2/\Gamma_\eta}\, \frac{{\rm i}\boldsymbol{k} \cdot \boldsymbol{B}_0}{\Gamma_\eta}. \tag{56}$$

We can use these solutions to calculate $\boldsymbol{\mathcal{E}}$. For getting an explicit expression, we also need the force correlation function in Fourier

[1] http://www.nordita.dk/software/pencil-code

space. For isotropic, homogeneous and statistically stationary forcing, this is given by

$$\overline{\tilde{f}_j(\boldsymbol{p},\omega)\ \tilde{f}_k(\boldsymbol{q},\omega')} = \delta^3(\boldsymbol{p}+\boldsymbol{q})\delta(\omega+\omega')F_{jk}(\boldsymbol{q},\omega), \quad (57)$$

where we can still take F_{jk} to be of the form given by equation (15), with spectral functions now changed to say, a frequency dependent $\bar{\Phi}(k,\omega)$ and $\bar{\chi}(k,\omega)$.

Note that in the limit where the correlation time of the forcing function is short (delta-correlated in time) the Fourier space spectral function F_{jk} is nearly independent of the frequency ω. However, if one evaluates the helicity of the forcing, one gets

$$\int \bar{\chi}(k,\omega)\ \mathrm{d}k\ \mathrm{d}\omega = \overline{\boldsymbol{f}\cdot(\nabla\times\boldsymbol{f})} \equiv H_{\mathrm{f}}, \quad (58)$$

where the k integral is from 0 to ∞, while the ω integration goes from $-\infty$ to $+\infty$. For $\bar{\chi}$ independent of ω this would be infinite. So we still keep a spectral dependence and write $\bar{\chi}(q,\omega) = \chi(q)g(\omega)$, where g is an even function of ω, satisfying $\int g(\omega)\mathrm{d}\omega = 1$. [The property that g is even is a consequence of the forcing function being real; see equation (7.44) of (Moffatt 1978).] For a forcing with say a correlation time τ, $g(\omega)$ will be nearly constant for $\omega\tau \sim 1$ and decay at large ω. In the limit of small τ the range for which $g(\omega)$ is nearly constant will be very large. We will need only $g(0) \sim \tau$ in most of what follows. Note that the other extreme limit of steady forcing corresponds to taking $g(\omega) \to \delta(\omega)$. The mean emf can be written as

$$\mathcal{E}_i(\boldsymbol{x},t) = \epsilon_{ijk}\overline{\boldsymbol{u}_j\boldsymbol{b}_k} = \int \tilde{\mathcal{E}}_i(\boldsymbol{k},\Omega)\ \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}-\mathrm{i}\Omega t}\mathrm{d}\boldsymbol{k}\ \mathrm{d}\Omega, \quad (59)$$

where the Fourier transform $\tilde{\boldsymbol{\mathcal{E}}}$ is given by

$$\tilde{\mathcal{E}}_i(\boldsymbol{k},\Omega) = \epsilon_{ijk}\int \overline{\tilde{u}_j(\boldsymbol{k}-\boldsymbol{q},\Omega-\omega)\tilde{b}_k(\boldsymbol{q},\omega)}\ \mathrm{d}\boldsymbol{q}\ \mathrm{d}\omega. \quad (60)$$

We now turn to the calculation of the non-linear mean emf and the resulting non-linear α-effect. We focus on Method A, the direct Method B and also Method C to illustrate the similarities and differences from the case when the time evolution is neglected. We also discuss in detail the result of applying a τ-approximation type method in the subsequent section.

4.1 Computing $\mathcal{E}$ from the induction equation

As before, we first start from the induction equation to solve for $\boldsymbol{b}$ in terms of $\boldsymbol{u}$ using equation (52), so

$$\tilde{\mathcal{E}}_i(\boldsymbol{k}) = \mathrm{i}\epsilon_{ijk}\int \boldsymbol{B}_0\cdot\boldsymbol{q}\ \frac{\overline{\tilde{u}_j(\boldsymbol{k}-\boldsymbol{q},\Omega-\omega)\tilde{u}_k(\boldsymbol{q},\omega)}}{-\mathrm{i}\omega+\eta q^2}\ \mathrm{d}\boldsymbol{q}\mathrm{d}\omega. \quad (61)$$

We can then express $\boldsymbol{u}$ in terms of $\boldsymbol{f}$ using equation (55). Substituting from equation (57) for the force correlation in a time-dependent flow, the mean emf in coordinate space is then given by

$$\mathcal{E}_i(\boldsymbol{x}) = \mathrm{i}\epsilon_{ijk}\int \frac{\boldsymbol{B}_0\cdot\boldsymbol{q}}{\Gamma_\eta}\frac{F_{jk}(\boldsymbol{q},\omega)}{|\Gamma_\nu+(\boldsymbol{B}_0\cdot\boldsymbol{q})^2/\Gamma_\eta|^2}\ \mathrm{d}\boldsymbol{q}\ \mathrm{d}\omega. \quad (62)$$

As usual, we define $\alpha = \boldsymbol{\mathcal{E}}\cdot\boldsymbol{B}_0/B_0^2$. Since only the antisymmetric part of F_{jk} contributes in the above integral, we have

$$\alpha = -\int \frac{(\hat{\boldsymbol{B}}_0\cdot\boldsymbol{q})^2\ \chi(q)\ g(\omega)\ (\mathrm{i}\omega+\eta q^2)\ \mathrm{d}\boldsymbol{q}\ \mathrm{d}\omega}{4\pi q^4\ |\ \Gamma_\nu\Gamma_\eta+(\boldsymbol{B}_0\cdot\boldsymbol{q})^2\ |^2}. \quad (63)$$

In the following we refer to this expression for α also as $\hat{\alpha}_{\mathrm{K}}$, since it is seen to arise purely from the $(\nabla^{-2}\ \boldsymbol{u}\times\boldsymbol{u})$-type velocity correlation, generalized to the time-dependent case. Note that the denominator of equation (63) is even in ω and so is the spectral function $g(\omega)$. Therefore the term in the above integral, which has an i ω in the numerator, vanishes on integration over ω (by symmetry). So the mean emf is a real quantity as it should be. Before evaluating the above integral explicitly, let us ask if we get the same expression for equation (62) using Methods B and C, even in the time-dependent case.

4.2 Computing $\mathcal{E}$ directly

Let us directly compute $\boldsymbol{\mathcal{E}} = \overline{\boldsymbol{u}\times\boldsymbol{b}}$ by substituting $\boldsymbol{u}$ and $\boldsymbol{b}$ in terms of $\boldsymbol{f}$, using equations (55) and (56). We also substitute from equation (57) for the force correlation in a time-dependent flow. The mean emf in coordinate space is then given by equation (62), so we do not repeat it here.

4.3 $\mathcal{E}$ from the momentum equation

As before we start from the momentum equation, solve for $\boldsymbol{u}$ in terms of $\boldsymbol{b}$ and the forcing function $\boldsymbol{f}$, and then substitute for $\boldsymbol{b}$ in terms of $\boldsymbol{f}$ using equation (56). We particularly wish to examine if the $(\nabla^{-2}\boldsymbol{f})\times\boldsymbol{b}$-like correlation is essential for calculating the $\boldsymbol{\mathcal{E}}$ correctly, even for the time-dependent case. Using equation (53) one can write

$$\tilde{u}_i(\boldsymbol{k},\omega) = \frac{1}{\Gamma_\nu}\left[(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{B}_0)\,\tilde{b}_i(\boldsymbol{k},\omega)+\tilde{f}_i(\boldsymbol{k},\omega)\right]. \quad (64)$$

From equation (60) the mean emf can then be written as

$$\begin{aligned}\tilde{\mathcal{E}}_i(\boldsymbol{k},\Omega) = {}& \epsilon_{ijk}\int \frac{\overline{\tilde{f}_j(\boldsymbol{k}-\boldsymbol{q},\Omega-\omega)\tilde{b}_k(\boldsymbol{q},\omega)}}{-\mathrm{i}(\Omega-\omega)+\nu(\boldsymbol{k}-\boldsymbol{q})^2}\ \mathrm{d}\boldsymbol{q}\ \mathrm{d}\omega \\ & +\mathrm{i}\epsilon_{ijk}\int \frac{\boldsymbol{B}_0\cdot(\boldsymbol{k}-\boldsymbol{q})}{-\mathrm{i}(\Omega-\omega)+\nu(\boldsymbol{k}-\boldsymbol{q})^2} \\ & \times\overline{\tilde{b}_j(\boldsymbol{k}-\boldsymbol{q},\Omega-\omega)\tilde{b}_k(\boldsymbol{q},\omega)}\ \mathrm{d}\boldsymbol{q}\ \mathrm{d}\omega.\end{aligned} \quad (65)$$

Here the first term involves the $(\nabla^{-2}\boldsymbol{f})\times\boldsymbol{b}$-like correlation, generalized to the time-dependent case. Substituting $\tilde{\boldsymbol{b}}$ in terms of $\tilde{\boldsymbol{f}}$ from equation (56) and integrating over the delta functions in wavenumbers and frequencies, the mean emf in coordinate space can be expressed as

$$\begin{aligned}\mathcal{E}_i(\boldsymbol{x}) = {}& \mathrm{i}\epsilon_{ijk}\int \frac{\boldsymbol{B}_0\cdot\boldsymbol{q}}{\Gamma_\eta}\frac{F_{jk}(\boldsymbol{q},\omega)}{|\Gamma_\nu|^2\ \left[1+\bar{N}\right]}\ \mathrm{d}\boldsymbol{q}\ \mathrm{d}\omega \\ & -\mathrm{i}\epsilon_{ijk}\int \frac{\boldsymbol{B}_0\cdot\boldsymbol{q}}{\Gamma_\eta}\frac{F_{jk}(\boldsymbol{q},\omega)}{|\Gamma_\nu|^2\ \left[1+\bar{N}\right]}\ \frac{\bar{N}^*}{1+\bar{N}^*}\ \mathrm{d}\boldsymbol{q}\ \mathrm{d}\omega.\end{aligned} \quad (66)$$

Here we have defined $\bar{N} = (\boldsymbol{B}_0\cdot\boldsymbol{q})^2/\Gamma_\eta\Gamma_\nu$. Now the limit of small Lorentz forces corresponds to taking $|\bar{N}| \ll 1$. Again in this limit the second integral vanishes while the first one [i.e. the generalized $(\nabla^{-2}\boldsymbol{f})\times\boldsymbol{b}$-like correlation in equation (66)] goes over to a kinematic α-effect. In fact, this part of the α-effect can be obtained from equations (55) and (56) by neglecting the Lorentz force in the expression for $\tilde{u}_i(\boldsymbol{k},\omega)$.

On adding the two terms in equation (66), the mean emf turns out to be identical to equation (62), obtained when starting from the induction equation or in the direct method. Therefore, one could either compute the mean emf starting from the induction equation, or directly, or from the momentum equation, by the addition of a generalized $(\nabla^{-2}\boldsymbol{f})\times\boldsymbol{b}$-like correlation and a purely magnetic correlation.

We can again define $\hat{\alpha}_{\mathrm{F}}$ and $\hat{\alpha}_{\mathrm{M}}$ for the time-dependent forcing, from the first and the second terms on the RHS of equation (66), respectively. We have

$$\hat{\alpha}_{\mathrm{F}} = -\int \frac{(\hat{\boldsymbol{B}}_0\cdot\boldsymbol{q})^2\ \chi(q)\ g(\omega)\left[\Gamma_\nu^*\Gamma_\eta^*+(\boldsymbol{B}_0\cdot\boldsymbol{q})^2\right]\ \mathrm{d}\boldsymbol{q}\ \mathrm{d}\omega}{4\pi q^4\ \Gamma_\nu^*\ |\ \Gamma_\nu\Gamma_\eta+(\boldsymbol{B}_0\cdot\boldsymbol{q})^2\ |^2}, \quad (67)$$

$$\hat{\alpha}_{\mathrm{M}} = \int \frac{B_0^2\,(\hat{\boldsymbol{B}}_0\cdot\boldsymbol{q})^4\,\chi(q)\,g(\omega)\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\omega}{4\pi q^4\,\Gamma_\nu^*\mid\Gamma_\nu\Gamma_\eta+(\boldsymbol{B}_0\cdot\boldsymbol{q})^2\mid^2}. \quad (68)$$

In terms of the total α, we have $\alpha = \hat{\alpha}_{\mathrm{F}} + \hat{\alpha}_{\mathrm{M}}$. It is now explicitly apparent that in the time-dependent case, $\alpha = \hat{\alpha}_{\mathrm{K}} = \hat{\alpha}_{\mathrm{F}} + \hat{\alpha}_{\mathrm{M}}$ in agreement with what is obtained for steady forcing. Further, in the limit of steady forcing where, $g(\omega) \to \delta(\omega)$, the above generalized expressions reduce to the $\hat{\alpha}$ values obtained from equations (25) and (31), respectively.

4.4 The non-linear α-effect

Let us return to the explicit computation of the non-linear α-effect for the delta-correlated flow. Solving for $\alpha = \boldsymbol{\mathcal{E}}\cdot\boldsymbol{B}_0/B_0^2$ from equation (63) leads to

$$\alpha = -\int \chi(q)\,(\hat{\boldsymbol{B}}_0\cdot\hat{\boldsymbol{q}})^2\,(\eta q^2)\,I(\boldsymbol{q})\,\frac{\mathrm{d}\boldsymbol{q}}{4\pi q^2}, \quad (69)$$

where I is the integral over ω given by

$$I = \int \frac{g(\omega)\mathrm{d}\omega}{|\Gamma_\nu\Gamma_\eta+(\boldsymbol{B}_0\cdot\boldsymbol{q})^2|^2} = \tau\int\frac{\mathrm{d}\omega}{|\Gamma_\nu\Gamma_\eta+(\boldsymbol{B}_0\cdot\boldsymbol{q})^2|^2}, \quad (70)$$

where the latter expression for I obtains in the limit of a delta-correlated forcing, where $g(\omega) = \tau$ is almost constant through the range where the rest of the integrand contributes significantly. We now focus on the special case of $\nu = \eta$ for the explicit evaluation of the above integral. Note that calculating I is then straightforward but tedious. We briefly outline the steps and then quote the result. First the denominator of equation (70) can be expanded and then factorized to give

$$|\Gamma_\nu\Gamma_\eta+(\boldsymbol{B}_0\cdot\boldsymbol{q})^2|^2 = (\omega+z)(\omega+z^*)(\omega-z)(\omega-z^*), \quad (71)$$

where $z = (\boldsymbol{B}_0\cdot\boldsymbol{q}) + \mathrm{i}\,\nu q^2$. Then the integral can be rewritten as

$$\int_{-\infty}^{\infty}\frac{\tau\mathrm{d}\omega}{2(z^{*2}-z^2)|z|^2}\left[\frac{z}{\omega-z^*}-\frac{z}{\omega+z^*}-\frac{z^*}{\omega-z}+\frac{z^*}{\omega+z}\right]. \quad (72)$$

Grouping the term having $\omega - z^*$ with the one having $\omega - z$ and likewise the two terms with $\omega + z^*$ and $\omega + z$, we get

$$I = \int_{-\infty}^{\infty}\frac{\tau\mathrm{d}\omega}{2(z^*+z)|z|^2} \times\left[\frac{-\omega+2\boldsymbol{B}_0\cdot\boldsymbol{q}}{(\omega-\boldsymbol{B}_0\cdot\boldsymbol{q})^2+\nu^2q^4}+\frac{\omega+2\boldsymbol{B}_0\cdot\boldsymbol{q}}{(\omega+\boldsymbol{B}_0\cdot\boldsymbol{q})^2+\nu^2q^4}\right], \quad (73)$$

where the last expression is explicitly real. A change of variables allows the above integral to be done easily to give [2]

$$I = \frac{\pi}{2}\frac{\tau}{\nu q^2[(\boldsymbol{B}_0\cdot\boldsymbol{q})^2+\nu^2q^4]}. \quad (74)$$

Substituting I into equation (69), and carrying out the angular integral over $\mu = \hat{\boldsymbol{B}}_0\cdot\hat{\boldsymbol{q}}$ then gives

$$\alpha = -\frac{\pi}{2}\tau\int_0^\infty\frac{\chi(q)}{\nu^2q^4}\,G(\beta)\,\mathrm{d}q, \quad (75)$$

where $\beta = B_0/\nu q$ for the $\nu = \eta$ case, and

$$G(\beta) = \frac{1}{\beta^2}\left(1-\frac{\tan^{-1}\beta}{\beta}\right). \quad (76)$$

So for the time-dependent delta-correlated flow, in the asymptotic limit of large B_0, we have $\alpha \to B_0^{-2}$. If we further assume that the forcing is at a particular wavenumber, q_0, and choose $\chi(q) = H_{\mathrm{f}}\delta(q-q_0)$, we have

$$\frac{\alpha}{\alpha_0} = 3\left(\frac{B_0}{B_{\mathrm{cr}}}\right)^{-2}\left[1-\frac{\tan^{-1}(B_0/B_{\mathrm{cr}})}{B_0/B_{\mathrm{cr}}}\right], \quad (77)$$

where we have now defined

$$\alpha_0 = -\frac{\pi}{6}\frac{\tau H_{\mathrm{f}}}{q_0^2B_{\mathrm{cr}}^2}, \quad B_{\mathrm{cr}} = \nu q_0. \quad (78)$$

[2] We thank K.-H. Rädler for pointing out to us that this result can be generalized for $\eta \neq \nu$, to give $I = \pi\tau/\{(\eta+\nu)q^2[(\boldsymbol{B}_0\cdot\boldsymbol{q})^2+\eta\nu q^4]\}$.

4.5 Comparison with τ-approximation

Note that in the case of time-dependent forcing, considered for example in the simulations of Brandenburg (2001) and Brandenburg & Subramanian (2005b), both $\dot{\boldsymbol{b}}$ and $\dot{\boldsymbol{u}}$ are finite, and hence also $\overline{\boldsymbol{u}\times\dot{\boldsymbol{b}}}$ and $\overline{\dot{\boldsymbol{u}}\times\boldsymbol{b}}$ can in general be finite, even though their sum would vanish in the statistically steady state. So taking a time derivative of $\boldsymbol{\mathcal{E}}$ and then examining the stationary situation, could break the degeneracy between $\hat{\alpha}_{\mathrm{K}}$ and $\hat{\alpha}_{\mathrm{F}}$ in the kinematic limit, and also lead to novel insights. Indeed, if one were not able to solve for $\boldsymbol{u}$ and $\boldsymbol{b}$ explicitly in terms of the forcing this would be the practical route to follow.

We now examine the time-dependent case in a manner analogous to the so-called τ-approximation. As mentioned earlier, the τ-approximation closures involve invoking a closure whereby triple correlations which arise during the evaluation of $\partial\boldsymbol{\mathcal{E}}/\partial t$, are assumed to provide a damping term proportional to $\boldsymbol{\mathcal{E}}$ itself. In the present context there is no need to invoke a closure for the triple correlations, because these terms are small for low fluid and magnetic Reynolds numbers. It turns out that the correct expression for $\boldsymbol{\mathcal{E}}$ can still be derived in the same framework, where one evaluates the $\partial\boldsymbol{\mathcal{E}}/\partial t$ expression. Further, it allows us to define a new set of α values for the time-dependent forcing, α_{K}, α_{M} and α_{F}, that is, without hat, which, respectively, incorporate the kinetic $(\boldsymbol{u}, \boldsymbol{u})$, magnetic $(\boldsymbol{b}, \boldsymbol{b})$ and the force–field $(\boldsymbol{f}, \boldsymbol{b})$ correlations (see below). These α values have properties very similar to those which arise in the τ-approximation closure for large R_{m} systems.

For example, we showed above that for steady forcing, the $\nabla^{-2}\boldsymbol{f}\times\boldsymbol{b}$ correlation is non-zero and essential to calculate the $\boldsymbol{\mathcal{E}}$ correctly in Method C. We also showed above that such a correlation is important to include even for a time-dependent forcing, if one starts with the explicit solution of the momentum equation for $\tilde{\boldsymbol{u}}$. On the other hand, in a τ-approximation type approach one first evaluates the $\partial\boldsymbol{\mathcal{E}}/\partial t$ expression, which involves a $\overline{\dot{\boldsymbol{u}}\times\boldsymbol{b}}$ type correlation, instead of solving first for $\boldsymbol{u}$. It is then interesting to ask, is the corresponding $(\boldsymbol{f}, \boldsymbol{b})$ correlation, or the α_{F} term defined below, which arises in the evaluation of $\overline{\dot{\boldsymbol{u}}\times\boldsymbol{b}}$, still non-zero for the time-dependent, delta-correlated forcing? Or does it vanish in the τ-approximation type approach, as assumed in earlier work (Blackman & Field 2002; Rädler et al. 2003; Brandenburg & Subramanian 2005a)? In particular, does one then recover $\alpha = \alpha_{\mathrm{K}} + \alpha_{\mathrm{M}}$, a relation which one obtains in the τ-approximation at large R_{m}? We examine these issues in detail below.

We write the time derivative of the emf as $\dot{\boldsymbol{\mathcal{E}}} = \dot{\boldsymbol{\mathcal{E}}}_{\mathrm{K}} + \dot{\boldsymbol{\mathcal{E}}}_{\mathrm{M}}$, where $\dot{\boldsymbol{\mathcal{E}}}_{\mathrm{K}} = \overline{\boldsymbol{u}\times\dot{\boldsymbol{b}}}$ and $\dot{\boldsymbol{\mathcal{E}}}_{\mathrm{M}} = \overline{\dot{\boldsymbol{u}}\times\boldsymbol{b}}$. From the induction equation for $\boldsymbol{b}$ and the momentum equation for $\boldsymbol{u}$, we now have

$$\dot{\boldsymbol{\mathcal{E}}}_{\mathrm{K}} = \overline{\boldsymbol{u}\times\dot{\boldsymbol{b}}} = \overline{\boldsymbol{u}\times\boldsymbol{B}_0\cdot\nabla\boldsymbol{u}} + \overline{\boldsymbol{u}\times\eta\nabla^2\boldsymbol{b}}, \quad (79)$$

$$\dot{\boldsymbol{\mathcal{E}}}_{\mathrm{M}} = \overline{\dot{\boldsymbol{u}}\times\boldsymbol{b}} = \overline{\boldsymbol{f}\times\boldsymbol{b}} + \overline{\boldsymbol{B}_0\cdot\nabla\boldsymbol{b}\times\boldsymbol{b}} + \overline{\nu\nabla^2\boldsymbol{u}\times\boldsymbol{b}}, \quad (80)$$

where the perturbed pressure term vanishes for divergence free forcing. We can evaluate each of these terms in Fourier space, since we have the Fourier space solutions for both $\tilde{\boldsymbol{u}}$ and $\tilde{\boldsymbol{b}}$ completely in terms of $\tilde{\boldsymbol{f}}$.

In the time-dependent case, we define by analogy to equation (33),

$$\boldsymbol{E}(\boldsymbol{k},\boldsymbol{q},\Omega,\omega)=\overline{\tilde{\boldsymbol{u}}(\boldsymbol{k}-\boldsymbol{q},\Omega-\omega)\times\tilde{\boldsymbol{b}}(\boldsymbol{q},\omega)}, \tag{81}$$

so that the emf in coordinate space is

$$\boldsymbol{\mathcal{E}}=\int\boldsymbol{E}(\boldsymbol{k},\boldsymbol{q},\Omega,\omega)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}-\mathrm{i}\Omega t}\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\Omega\,\mathrm{d}\omega. \tag{82}$$

We also define, analogous to equations (34) and (35),

$$\boldsymbol{E}_{\mathrm{K}}^{(t)}(\boldsymbol{k},\boldsymbol{q},\Omega,\omega)=\overline{\tilde{\boldsymbol{u}}(\boldsymbol{k}-\boldsymbol{q},\Omega-\omega)\times[-\mathrm{i}\omega\tilde{\boldsymbol{b}}(\boldsymbol{q},\omega)]}, \tag{83}$$

$$\boldsymbol{E}_{\mathrm{M}}^{(t)}(\boldsymbol{k},\boldsymbol{q},\Omega,\omega)=\overline{[-\mathrm{i}(\Omega-\omega)\tilde{\boldsymbol{u}}(\boldsymbol{k}-\boldsymbol{q},\Omega-\omega)]\times\tilde{\boldsymbol{b}}(\boldsymbol{q},\omega)}. \tag{84}$$

Note that $\boldsymbol{E}_{\mathrm{K}}^{(t)}+\boldsymbol{E}_{\mathrm{M}}^{(t)}=-\mathrm{i}\,\Omega\boldsymbol{E}$. Using the induction and momentum equations, we have explicitly,

$$\boldsymbol{E}_{\mathrm{K}}^{(t)}=\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{B}_0\,[\overline{\tilde{\boldsymbol{u}}(\boldsymbol{k}-\boldsymbol{q},\Omega-\omega)\times\tilde{\boldsymbol{u}}(\boldsymbol{q},\omega)}]-\eta q^2\boldsymbol{E}, \tag{85}$$

$$\begin{aligned}\boldsymbol{E}_{\mathrm{M}}^{(t)}=&\ \mathrm{i}(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{B}_0\,[\overline{\tilde{\boldsymbol{b}}(\boldsymbol{k}-\boldsymbol{q},\Omega-\omega)\times\tilde{\boldsymbol{b}}(\boldsymbol{q},\omega)}]\\&+\overline{\tilde{\boldsymbol{f}}(\boldsymbol{k}-\boldsymbol{q},\Omega-\omega)\times\tilde{\boldsymbol{b}}(\boldsymbol{q},\omega)}-\nu(\boldsymbol{k}-\boldsymbol{q})^2\boldsymbol{E}.\end{aligned} \tag{86}$$

We add equations (85) and (86), to write

$$\begin{aligned}-\mathrm{i}\Omega\boldsymbol{E}+\frac{\boldsymbol{E}}{\tau_{\mathrm{eff}}}=&\ \mathrm{i}\boldsymbol{q}\cdot\boldsymbol{B}_0\Phi(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{u}})+\mathrm{i}(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{B}_0\Phi(\tilde{\boldsymbol{b}},\tilde{\boldsymbol{b}})\\&+\Phi(\tilde{\boldsymbol{f}},\tilde{\boldsymbol{b}}),\end{aligned} \tag{87}$$

where we have defined, for any pair of vector fields $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$,

$$\Phi(\tilde{\boldsymbol{f}}_1,\tilde{\boldsymbol{f}}_2)=\overline{\tilde{\boldsymbol{f}}_1(\boldsymbol{k}-\boldsymbol{q},\Omega-\omega)\times\tilde{\boldsymbol{f}}_2(\boldsymbol{q},\omega)}, \tag{88}$$

and $\tau_{\mathrm{eff}}^{-1}=\eta q^2+\nu\,(\boldsymbol{k}-\boldsymbol{q})^2$. We note in passing that the above equation is similar to the corresponding equation which obtains under τ-approximation in the large Reynolds number case, except that τ_{eff} would then correspond to a relaxation time for triple correlations (cf. Brandenburg & Subramanian 2005a). So we have

$$\boldsymbol{E}=\tau_{\mathrm{eff}}^{*}\left[\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{B}_0\Phi(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{u}})+\mathrm{i}(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{B}_0\Phi(\tilde{\boldsymbol{b}},\tilde{\boldsymbol{b}})+\Phi(\tilde{\boldsymbol{f}},\tilde{\boldsymbol{b}})\right], \tag{89}$$

where we define $\tau_{\mathrm{eff}}^{*}=\tau_{\mathrm{eff}}/[1-\mathrm{i}\,\Omega\,\tau_{\mathrm{eff}}]$. Let us define $\alpha=(\boldsymbol{\mathcal{E}}\cdot\boldsymbol{B}_0)/B_0^2$ as before. Then in coordinate space,

$$\alpha=\alpha_{\mathrm{K}}+\alpha_{\mathrm{M}}+\alpha_{\mathrm{F}}, \tag{90}$$

where

$$\alpha_{\mathrm{K}}=\int\mathrm{i}\boldsymbol{q}\cdot\hat{\boldsymbol{B}}_0\,\Phi(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{u}})\cdot\hat{\boldsymbol{B}}_0\,\tau_{\mathrm{eff}}^{*}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}-\mathrm{i}\Omega t}\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\Omega\,\mathrm{d}\omega, \tag{91}$$

$$\alpha_{\mathrm{M}}=\int\mathrm{i}(\boldsymbol{k}-\boldsymbol{q})\cdot\hat{\boldsymbol{B}}_0\,\Phi(\tilde{\boldsymbol{b}},\tilde{\boldsymbol{b}})\cdot\hat{\boldsymbol{B}}_0\,\tau_{\mathrm{eff}}^{*}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}-\mathrm{i}\Omega t}\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\Omega\,\mathrm{d}\omega, \tag{92}$$

$$\alpha_{\mathrm{F}}=\frac{1}{B_0}\int\Phi(\tilde{\boldsymbol{f}},\tilde{\boldsymbol{b}})\cdot\hat{\boldsymbol{B}}_0\,\tau_{\mathrm{eff}}^{*}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}-\mathrm{i}\Omega t}\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\Omega\,\mathrm{d}\omega \tag{93}$$

correspond, respectively, to the terms containing the $(\boldsymbol{u},\boldsymbol{u})$, $(\boldsymbol{b},\boldsymbol{b})$ and $(\boldsymbol{f},\boldsymbol{b})$ correlations on the RHS of equation (89).

Substituting $\tilde{\boldsymbol{u}}$ and $\tilde{\boldsymbol{b}}$ in terms of $\tilde{\boldsymbol{f}}$ from equations (55) and (56), and integrating over the delta functions in wavenumbers and frequencies which arises in taking the $(\tilde{\boldsymbol{f}},\tilde{\boldsymbol{f}})$ correlations, we then have in the coordinate space,

$$\alpha_{\mathrm{K}}=-\int\frac{(\hat{\boldsymbol{B}}_0\cdot\boldsymbol{q})^2\,\chi(q)\,g(\omega)\mid\Gamma_\eta\mid^2\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\omega}{4\pi q^4\mid\Gamma_\nu\Gamma_\eta+(\boldsymbol{B}_0\cdot\boldsymbol{q})^2\mid^2(\eta+\nu)q^2}, \tag{94}$$

$$\alpha_{\mathrm{M}}=\int\frac{B_0^2(\hat{\boldsymbol{B}}_0\cdot\boldsymbol{q})^4\,\chi(q)\,g(\omega)\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\omega}{4\pi q^4\mid\Gamma_\nu\Gamma_\eta+(\boldsymbol{B}_0\cdot\boldsymbol{q})^2\mid^2(\eta+\nu)q^2}, \tag{95}$$

$$\alpha_{\mathrm{F}}=-\int\frac{(\hat{\boldsymbol{B}}_0\cdot\boldsymbol{q})^2\,\chi(q)\,g(\omega)\left[\Gamma_\nu^*\Gamma_\eta^*+(\boldsymbol{B}_0\cdot\boldsymbol{q})^2\right]\mathrm{d}\boldsymbol{q}\,\mathrm{d}\omega}{4\pi q^4\mid\Gamma_\nu\Gamma_\eta+(\boldsymbol{B}_0\cdot\boldsymbol{q})^2\mid^2(\eta+\nu)q^2}. \tag{96}$$

On adding equations (94), (95) and (96), the expression for α is

$$\alpha=-\int\frac{(\hat{\boldsymbol{B}}_0\cdot\boldsymbol{q})^2\,\chi(q)\,g(\omega)\,I_\alpha}{4\pi q^4\,(\eta+\nu)q^2}\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\omega, \tag{97}$$

where

$$I_\alpha=\frac{\mid\Gamma_\eta\mid^2-(\boldsymbol{B}_0\cdot\boldsymbol{q})^2+\Gamma_\nu^*\Gamma_\eta^*+(\boldsymbol{B}_0\cdot\boldsymbol{q})^2}{\mid\Gamma_\nu\Gamma_\eta+(\boldsymbol{B}_0\cdot\boldsymbol{q})^2\mid^2}. \tag{98}$$

The numerator of this integrand can be simplified to give $\Gamma_\eta^*[\eta q^2+\nu q^2]$ such that

$$\alpha=-\frac{1}{B_0^2}\int\frac{(\boldsymbol{B}_0\cdot\boldsymbol{q})^2\,\chi(q)\,g(\omega)\,\Gamma_\eta^*}{4\pi q^4\mid\Gamma_\nu\Gamma_\eta+(\boldsymbol{B}_0\cdot\boldsymbol{q})^2\mid^2}\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\omega. \tag{99}$$

It is apparent from equation (99) that the expression for α turns out to be the same as in equation (63). Therefore, the τ-approximation type treatment also gives the same α as the other methods.

It is interesting to consider what happens to the various α values defined in equations (94)–(97), in the limit of steady forcing, where $g(\omega)\to\delta(\omega)$. It is straightforward to check that in this steady forcing limit α defined in equation (99) goes over exactly to the total $\alpha=\alpha^{(\mathrm{S})}$ given by the steady state expression in equation (41) of Section 3.5. Also, in the steady forcing limit, we get

$$\alpha_{\mathrm{K}}\to[\eta/(\eta+\nu)]\hat{\alpha}_{\mathrm{K}}=[\eta/(\eta+\nu)]\alpha^{(\mathrm{S})}, \tag{100}$$

$$\alpha_{\mathrm{M}}\to[\nu/(\eta+\nu)]\hat{\alpha}_{\mathrm{M}}, \tag{101}$$

$$\alpha_{\mathrm{F}}\to[\nu/(\eta+\nu)]\hat{\alpha}_{\mathrm{F}}. \tag{102}$$

In this limit one has therefore

$$\alpha_{\mathrm{M}}+\alpha_{\mathrm{F}}=[\nu/(\eta+\nu)](\hat{\alpha}_{\mathrm{M}}+\hat{\alpha}_{\mathrm{F}})=[\nu/(\eta+\nu)]\alpha^{(\mathrm{S})} \tag{103}$$

and so, once again,

$$\alpha_{\mathrm{K}}+\alpha_{\mathrm{M}}+\alpha_{\mathrm{F}}=\alpha^{(\mathrm{S})}, \tag{104}$$

as expected. It should be emphasized, however, that for a general time-dependent forcing, there is no simple relation of the form given by the expressions (100)–(102).

Let us consider the case of delta-correlated forcing now in more detail. It is of interest to check if the α_{F} term contributes in the τ-approximation type closures, as it does in the time-independent case. We have from equation (96),

$$\alpha_{\mathrm{F}}=-\int\frac{\chi(q)\,(\hat{\boldsymbol{B}}_0\cdot\hat{\boldsymbol{q}})^2}{(\eta+\nu)q^2}\,I_{\mathrm{F}}(\boldsymbol{q})\,\frac{\mathrm{d}\boldsymbol{q}}{4\pi q^2}, \tag{105}$$

where I_{F} is the integral over ω given by

$$I_{\mathrm{F}}=\int\frac{\Gamma_\nu^*\Gamma_\eta^*+(\boldsymbol{B}_0\cdot\boldsymbol{q})^2}{|\Gamma_\nu\Gamma_\eta+(\boldsymbol{B}_0\cdot\boldsymbol{q})^2|^2}\,g(\omega)\,\mathrm{d}\omega. \tag{106}$$

Here we can simplify $\Gamma_\nu^*\Gamma_\eta^*+(\boldsymbol{B}_0\cdot\boldsymbol{q})^2=-\omega^2+\nu\eta q^4+(\boldsymbol{B}_0\cdot\boldsymbol{q})^2+\mathrm{i}\omega(\eta q^2+\nu q^2)$. The integral over the term odd in ω again vanishes, leaving again a real I_{F},

$$I_{\mathrm{F}}=\int\frac{-\omega^2+\nu\eta q^4+(\boldsymbol{B}_0\cdot\boldsymbol{q})^2}{|\Gamma_\nu\Gamma_\eta+(\boldsymbol{B}_0\cdot\boldsymbol{q})^2|^2}\,g(\omega)\,\mathrm{d}\omega. \tag{107}$$

Let us focus on the case $\eta=\nu$ as before. We rewrite the numerator using the identity $-\omega^2+\nu\eta q^4+(\boldsymbol{B}_0\cdot\boldsymbol{q})^2=-(\omega+z)(\omega-z^*)+\omega(z-z^*)$, so we have

$$I_{\mathrm{F}}=\int\frac{-(\omega+z)(\omega-z^*)+\omega(z-z^*)}{|\Gamma_\nu\Gamma_\eta+(\boldsymbol{B}_0\cdot\boldsymbol{q})^2|^2}\,g(\omega)\,\mathrm{d}\omega. \tag{108}$$

The second term in the numerator of equation (108) does not contribute to the integral, since it is odd in ω, while the denominator is even. To simplify the integral further we use the identity in equation (71) for its denominator, giving

$$
\begin{aligned}
I_{\rm F} &= -\int_{-\infty}^{\infty} \frac{g(\omega)\,{\rm d}\omega}{(\omega+z^*)(\omega-z)} \\
&= -\int_{-\infty}^{\infty} \frac{g(\omega)\,{\rm d}\omega}{z+z^*}\left[\frac{1}{\omega-z}-\frac{1}{\omega+z^*}\right] \\
&= -\int_{-\infty}^{\infty} \frac{g(\omega)\,{\rm d}\omega}{z+z^*}\left[\frac{(\omega-x)+{\rm i}y}{(\omega-x)^2+y^2}-\frac{(\omega+x)+{\rm i}y}{(\omega+x)^2+y^2}\right], \qquad (109)
\end{aligned}
$$

where we have defined $x = \boldsymbol{B}_0 \cdot \boldsymbol{q}$ and $y = \nu q^2$, which are the real and imaginary parts, respectively, of z. Now changing variables to $u = \omega - x$ in the first term and $u = \omega + x$ in the second term we see that

$$
I_{\rm F} = -\int_{-\infty}^{\infty} \frac{u+{\rm i}y}{z+z^*}\ \frac{g(u+x)-g(u-x)}{u^2+y^2}\,{\rm d}u = 0. \qquad (110)
$$

Note that $I_{\rm F} \to 0$ in the limit of a delta-correlated forcing, where $g(\omega) = \tau$ is almost constant through the range where the rest of the integrand contributes significantly; that is $g(u+x) = g(u-x) = \tau$ where the integrand contributes significantly, while $g(u+x) \to 0$ and $g(u-x) \to 0$ at large u. This can also be checked by doing the integral for $I_{\rm F}$ numerically. So, interestingly, $\alpha_{\rm F} = 0$. Thus, for a forcing which is random and delta-correlated in time, there is no contribution from the $\overline{\boldsymbol{f} \times \boldsymbol{b}}$ type correlation! Thus, α is the sum of just two terms, a kinetic and a magnetic contribution which can be shown explicitly as follows.

We note that equation (94) can be expressed as

$$
\alpha_{\rm K} = -\frac{1}{4\pi(\eta+\nu)B_0^2}\int(\boldsymbol{B}_0\cdot\boldsymbol{q})^2\ \frac{\chi(q)}{q^6}\ I_{\rm K}(\boldsymbol{q})\,{\rm d}\boldsymbol{q}, \qquad (111)
$$

where

$$
\begin{aligned}
I_{\rm K} &= \tau\int\frac{\omega^2+\eta^2q^4}{|\ \Gamma_\nu\Gamma_\eta+(\boldsymbol{B}_0\cdot\boldsymbol{q})^2\ |^2}\,{\rm d}\omega \\
&= \tau\int\frac{\omega^2+\eta^2q^4+(\boldsymbol{B}_0\cdot\boldsymbol{q})^2-(\boldsymbol{B}_0\cdot\boldsymbol{q})^2}{|\ \Gamma_\nu\Gamma_\eta+(\boldsymbol{B}_0\cdot\boldsymbol{q})^2\ |^2}\,{\rm d}\omega. \qquad (112)
\end{aligned}
$$

As before, we focus on the case $\eta = \nu$ when the numerator can be simplified as $\omega^2 + \eta^2\, q^4 + (\boldsymbol{B}_0 \cdot \boldsymbol{q})^2 - (\boldsymbol{B}_0 \cdot \boldsymbol{q})^2 = (\omega + z)(\omega - z^*) - \omega(z + z^*) - (\boldsymbol{B}_0 \cdot \boldsymbol{q})^2$. So we have

$$
I_{\rm K} = \tau\int\frac{(\omega+z)(\omega-z^*)-\omega(z+z^*)-(\boldsymbol{B}_0\cdot\boldsymbol{q})^2}{|\ \Gamma_\nu\Gamma_\eta+(\boldsymbol{B}_0\cdot\boldsymbol{q})^2\ |^2}\,{\rm d}\omega. \qquad (113)
$$

It is to be noted that the second term in the squared bracket in equation (113) does not contribute to the integral. Using the identity in equation (71) for its denominator, we have

$$
\begin{aligned}
I_{\rm K} &= \tau\int\frac{{\rm d}\omega}{(\omega-z)(\omega-z^*)} - \tau\int\frac{{\rm d}\omega}{(\omega+z)(\omega+z^*)(\omega-z)(\omega-z^*)} \\
&= \frac{\pi\tau}{\nu q^2} - \frac{\pi\tau}{2\nu q^2}\ \frac{(\boldsymbol{B}_0\cdot\boldsymbol{q})^2}{(\boldsymbol{B}_0\cdot\boldsymbol{q})^2+\nu^2q^4}. \qquad (114)
\end{aligned}
$$

Substituting $I_{\rm K}$ into equation (111) and carrying out the angular integral over $\mu = \hat{\boldsymbol{B}}_0 \cdot \hat{\boldsymbol{q}}$ then gives

$$
\alpha_{\rm K} = -\frac{\pi}{6}\tau\int\frac{\chi(q)}{\nu^2q^4}\,{\rm d}\boldsymbol{q} + \frac{\pi}{4}\tau B_0^2\int\frac{\chi(q)}{\nu^2q^4}\ H(\beta)\,{\rm d}\boldsymbol{q}, \qquad (115)
$$

where $\beta = B_0/\nu q$ for the $\nu = \eta$ case, and

$$
H(\beta) = \frac{1}{\beta^2}\left[\frac{1}{3}-\frac{1}{\beta^2}\left(1-\frac{\tan^{-1}\beta}{\beta}\right)\right]. \qquad (116)
$$

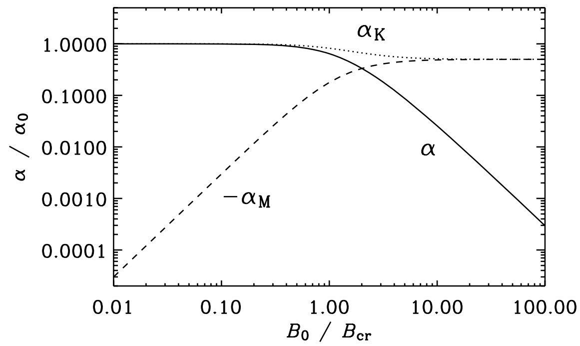

Figure 4. Variation of α, $\alpha_{\rm K}$ and $-\alpha_{\rm M}$ with B_0 from the analytical theory using a delta correlated forcing. Note that $\alpha = \alpha_{\rm K} + \alpha_{\rm M}$ and $\alpha_{\rm F} = 0$.

A similar analysis for equation (95) yields,

$$
\alpha_{\rm M} = \frac{1}{8\pi\nu B_0^2}\int(\boldsymbol{B}_0\cdot\boldsymbol{q})^2\ \frac{\chi(q)}{q^6}\ I_{\rm M}(\boldsymbol{q})\,{\rm d}\boldsymbol{q}, \qquad (117)
$$

where

$$
\begin{aligned}
I_{\rm M} &= \tau\int\frac{(\boldsymbol{B}_0\cdot\boldsymbol{q})^2\,{\rm d}\omega}{|\ \Gamma_\nu\Gamma_\eta+(\boldsymbol{B}_0\cdot\boldsymbol{q})^2\ |^2} \\
&= \frac{\pi\tau}{2\nu q^2}\ \frac{(\boldsymbol{B}_0\cdot\boldsymbol{q})^2}{(\boldsymbol{B}_0\cdot\boldsymbol{q})^2+\nu^2q^4}. \qquad (118)
\end{aligned}
$$

Carrying out the angular integral as earlier gives

$$
\alpha_{\rm M} = \frac{\pi}{4}\tau B_0^2\int\frac{\chi(q)}{\nu^2q^4}\ H(\beta)\,{\rm d}\boldsymbol{q}. \qquad (119)
$$

If we further assume that the forcing is at a particular wavenumber, q_0, and choose $\chi(q) = H_{\rm f}\delta(q - q_0)$, we then have

$$
\frac{\alpha_{\rm K}}{\alpha_0} = \frac{1}{2} + \frac{3}{2}\left(\frac{B_0}{B_{\rm cr}}\right)^{-2}\left[1-\frac{\tan^{-1}(B_0/B_{\rm cr})}{B_0/B_{\rm cr}}\right], \qquad (120)
$$

$$
\frac{\alpha_{\rm M}}{\alpha_0} = -\frac{1}{2} + \frac{3}{2}\left(\frac{B_0}{B_{\rm cr}}\right)^{-2}\left[1-\frac{\tan^{-1}(B_0/B_{\rm cr})}{B_0/B_{\rm cr}}\right], \qquad (121)
$$

where α_0 and $B_{\rm cr}$ were defined in equation (78). It is explicitly apparent that $\alpha = \alpha_{\rm K} + \alpha_{\rm M}$, in agreement with equation (77). The result is plotted in Fig. 4. Note also that $\alpha_{\rm F} = 0$, as was assumed in the minimal τ-approximation (MTA) type calculations for large fluid and magnetic Reynolds numbers (Brandenburg & Subramanian 2005a).

It is interesting to note that in the limit $B_0/B_{\rm cr} \gg 1$, $\alpha_{\rm K} \to +\alpha_0/2 + O(B_0^{-2})$ and $\alpha_{\rm M} \to -\alpha_0/2 + O(B_0^{-2})$ and so the total $\alpha = \alpha_{\rm K} + \alpha_{\rm M} \to 0$ as B_0^{-2}. This is reminiscent of the kinetic and magnetic α values nearly cancelling to leave a small residual α-effect in EDQNM or MTA type closures. It is also interesting to consider the limit when $B_0/B_{\rm cr} \ll 1$. In this limit $\alpha_{\rm K} \to \alpha_0$ and $\alpha_{\rm M} \to 0$, and so the net α-effect is just the kinetic contribution. Finally, for any B_0 we note that $(\alpha_{\rm K} - \alpha_{\rm M})/\alpha_0 = 1$.

4.6 Comparison with simulations

It is appropriate to compare with the simulations of Brandenburg & Subramanian (2005b). We have produced additional results for low fluid and magnetic Reynolds numbers ($Re = R_{\rm m} \approx 2 \times 10^{-2}$). The forcing consists of helical waves with average wavenumber $k_{\rm f}/k_1 = 1.5$. The resulting values of α are fluctuating strongly, so it is important to average them in time. Instead of calculating the full integral expressions, we estimate the contributions to α from the

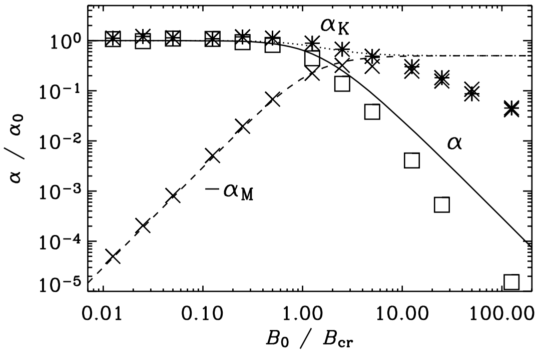

Figure 5. Variation of α, α_K and $-\alpha_M$ with B_0 for a random flow at low fluid and magnetic Reynolds numbers ($Re = R_m \approx 2 \times 10^{-2}$), compared with the analytic theory predicted for a fully isotropic flow. Note that the numerically determined values of α_K are smaller and those of $-\alpha_M$ larger than the corresponding analytic values, which is similar to the results for the ABC-flow forcing.

formulae $\alpha_K = -2\tau\langle u_x u_{z,y}\rangle$, $\alpha_M = 2\tau\langle b_x b_{z,y}\rangle$, $\alpha_F = \tau\langle \boldsymbol{f} \times \boldsymbol{b}\rangle \cdot \boldsymbol{B}_0/B_0^2$, where $\boldsymbol{B}_0 = (0, B_0, 0)$ is the imposed field, and $\tau^{-1} = (\nu + \eta)k_f^2$.

The result is shown in Fig. 5 and compared with the results of the previous section. In all cases the resulting values of α_F are negligibly small and will not be considered further. Like in the case of the ABC flow, the numerically estimated values of α are smaller than the analytic ones. This might be explicable if for some reason the relevant normalization in terms of α_0 were to depend on B_0. Alternatively, the discrepancy might be due to us using only simplified expressions instead of the full integral expressions. However, the important point is that the main contribution to the quenching comes from the growing contribution of $-\alpha_M$ such that $\alpha_K + \alpha_M$ is quenched to values much smaller than α_K. The corresponding results in the case of larger R_m are given by Brandenburg & Subramanian (2005b).

5 DISCUSSION

We have considered here the non-linear α-effect in the limit of small magnetic and fluid Reynolds numbers, for both steady and time-dependent (both for general and delta-correlated) forcings. In the limit of low R_m and Re, one can neglect terms non-linear in the fluctuating fields and hence explicitly solve for the small-scale magnetic and velocity fields using double FOSA. We can then calculate the α-effect in several different ways.

For both steady and time-dependent forcings, one gets similar results, provided one starts from the explicit solutions to the induction and momentum equations. Let us begin with a summary of the results for the steady forcing case. To begin with we follow in Method A, the traditional route of solving the induction equation for $\boldsymbol{b}$ in terms of $\boldsymbol{u}$, and then calculating the α-effect. For statistically isotropic velocity fields, this gives α dependent on the helicity of the velocity potential, as already known from previous work. In addition since we have an explicit solution for $\boldsymbol{u}$ in terms of $\boldsymbol{f}$, one can relate α directly to the helical part of the force correlation. In Method B we solved for $\boldsymbol{u}$ and $\boldsymbol{b}$ in terms of the forcing function $\boldsymbol{f}$ and computed $\boldsymbol{\mathcal{E}}$ directly. This would correspond to what is done when the α-effect is determined from simulations. However, in general this cannot be done analytically unless one can solve for the small-scale velocity and magnetic field explicitly. We get α identical to that obtained in Method A.

More interesting is Method C, where one takes the momentum equation as the starting point, instead of the induction equation. In the limit of small fluid Reynolds numbers one can solve for $\boldsymbol{u}$ in terms of $\boldsymbol{b}$ and hence compute $\boldsymbol{\mathcal{E}}$. This necessarily involves also the $(\nabla^{-2}\boldsymbol{f}) \times \boldsymbol{b}$ correlation, between the forcing and the small-scale magnetic field, in addition to the $(\nabla^{-2}\boldsymbol{j}) \cdot \boldsymbol{b}$ (or $\boldsymbol{a} \cdot \boldsymbol{b}$) correlation arising from the Lorentz force. This second term depends on the helicity of the small-scale magnetic fields. When the Lorentz force is small, the first term contributes to α in a manner closely related to the usual kinematic α-effect, while the second term contributes negligibly. Interestingly, as the Lorentz force gains in importance the first term is suppressed, while the second term (which has an opposite sign) gains in importance and cancels the first term, to further suppress the total α-effect (Fig. 1). This is similar to the suppression of the kinetic alpha due to the addition of a magnetic alpha (proportional to the helical part of $\boldsymbol{b}$) found in several closure models (Pouquet et al. 1976; Kleeorin & Ruzmaikin 1982; Gruzinov & Diamond 1994; Blackman & Field 2002; Brandenburg & Subramanian 2005b). When one combines the two terms, the resulting α-effect is identical to that obtained from the induction equation (Method A) using FOSA or the direct computation of Method B. However, it also highlights the fact that in this steady case the $(\nabla^{-2}\boldsymbol{f}) \times \boldsymbol{b}$-type correlation does not vanish, and that there is no tendency for this term to balance the viscous term, as one might have expected.

Finally, the results of Method D show that the formalism used in the τ-approximation lead to results that are equivalent to the usual approach taken in the FOSA. However, this requires that the detailed spectral dependence of the diffusion operator be retained until the point where the steady state assumption is made. The resulting equations are solved for the spectral electromotive force of the form $\boldsymbol{E}(\boldsymbol{k}, \boldsymbol{q})$ in equation (39). Otherwise, one would not recover the correct low-conductivity limit, as shown by Rädler & Rheinhardt (2007). We emphasize that throughout this paper we have understood the term τ-approximation only in this more generalized sense.

The above results are also obtained for statistically stationary but time-dependent forcing. Specifically, we showed that even in the time-dependent case, one gets identical results for the mean emf if α is computed directly (as in Method B), or from the momentum equation, by the addition of a generalized $(\nabla^{-2}\boldsymbol{f}) \times \boldsymbol{b}$-like correlation and a purely magnetic correlation. The explicit form of the α-effect differs between delta-correlated and steady forcing cases. In particular, in the limit of large B_0, and when $\nu = \eta$, we have $\alpha \propto B_0^{-2}$ for delta-correlated forcing, in contrast to $\alpha \propto B_0^{-3}$, for the case of a steady forcing. The former result has already been obtained by Field, Blackman & Chou (1999) and Rogachevskii & Kleeorin (2000), both of whom assumed the force–field correlation to vanish. However, their result was derived under the assumption of large fluid and magnetic Reynolds numbers.

The major difference between the time-dependent and steady forcing cases arise when one follows Method D, the formalism used in the τ-approximation type approaches to computing α in large R_m systems. We recall that in this approach one starts by evaluating the time derivative of the emf, and then look at the stationary limit. We showed that in this limit the α-effect can be naturally written as the sum of three terms, $\alpha = \alpha_K + \alpha_M + \alpha_F$, for a general time-dependent forcing. Here α_K and α_M are the kinetic and magnetic contributions to α corresponding, respectively, to the terms containing the $(\boldsymbol{u}, \boldsymbol{u})$ and $(\boldsymbol{b}, \boldsymbol{b})$ (see equations 91 and 92), while α_F incorporates the $(\boldsymbol{f}, \boldsymbol{b})$ correlation; see equation (93). Interestingly, we showed that $\alpha_F = 0$ in the approach of Method D, for delta-correlated forcing, and therefore $\alpha = \alpha_K + \alpha_M$, just the sum of a kinetic and magnetic

Table 1. Summary of the results obtained from FOSA and τ-approximation type analysis for steady and random forcings.

	Steady forcing	Delta-correlated forcing
FOSA	$\alpha = \hat{\alpha}_{\rm K}$ $= \hat{\alpha}_{\rm F} + \hat{\alpha}_{\rm M}$	$\alpha = \hat{\alpha}_{\rm K}$ $= \hat{\alpha}_{\rm F} + \hat{\alpha}_{\rm M}$
τ-approximation	$\alpha = \alpha_{\rm K} + \alpha_{\rm F} + \alpha_{\rm M}$	$\alpha = \alpha_{\rm K} + \alpha_{\rm F} + \alpha_{\rm M}$ $= \alpha_{\rm K} + \alpha_{\rm M}$

terms. We also computed $\alpha_{\rm K}$ and $\alpha_{\rm M}$ explicitly for the case $\eta = \nu$. In the kinematic limit, $\alpha_{\rm M} \to 0$, while $\alpha \to \alpha_{\rm K}$. While in the limit $B_0/B_{\rm cr} \gg 1$, $\alpha_{\rm K} \to +\alpha_0/2 + O(B_0^{-2})$ and $\alpha_{\rm M} \to -\alpha_0/2 + O(B_0^{-2})$, so that the total $\alpha = \alpha_{\rm K} + \alpha_{\rm M} \to 0$ as B_0^{-2}. This is reminiscent of the kinetic and magnetic α values nearly cancelling to leave a small residual α-effect in EDQNM or τ-approximation type closures. The results from employing FOSA and τ-approximation type analysis is summarized in Table 1 both for steady and random forcings.

As far as the low Reynolds number case is concerned, our analytic solutions demonstrate quite clearly that one can look at the non-linear α-effect in several equivalent ways. On one hand, one can express α completely in terms of the helical properties of the velocity field (Method A) as advocated by Proctor (2003) and Rädler & Rheinhardt (2007). At the same time α can be naturally expressed as a sum of a suppressed kinetic part (first term in Method C) and an oppositely signed magnetic part proportional to the helical part of $\boldsymbol{b}$ (second term in Method C).

Method D applied to the delta-correlated forcing is particularly revealing. As here one can explicitly write $\alpha = \alpha_{\rm K} + \alpha_{\rm M}$, or as the sum of a kinetic $\alpha_{\rm K}$, which dominates in the linear regime, and a magnetic $\alpha_{\rm M}$, which gains in importance as the field becomes stronger, and cancels $\alpha_{\rm K}$ to suppress the net α-effect. This is similar to the approach that arises from closure models like EDQNM (Pouquet et al. 1976) and the τ-approximation (Kleeorin et al. 1996; Blackman & Field 2000; Brandenburg & Subramanian 2005a) or the quasi-linear models (Gruzinov & Diamond 1994). In all these cases the non-linear α-effect, for large $R_{\rm m}$, is the sum of a kinetic part and an oppositely signed magnetic part. As noted above, the kinetic α-effect is itself suppressed, as seen in Fig. 4, but this happens only for $B_0 > B_{\rm cr}$, and the suppression is milder than the strong suppression of the total α-effect. This is also borne out in the simulations of Brandenburg & Subramanian (2005b), where the kinetic part of the α-effect is suppressed in a manner that is independent of the magnetic Reynolds number, even though the total α is catastrophically suppressed. Finally, we also have shown that $\alpha_{\rm F}$ defined naturally in the approach of Method D, vanishes for delta-correlated forcing, as was assumed in derivations of the α-effect in τ-approximation type closures.

In the special case of periodic domains it is now clear that for forced turbulence at large $R_{\rm m}$ the steady state α-effect is catastrophically quenched (Cattaneo & Hughes 1996; Brandenburg 2001). However, the physical cause of this phenomenon was long controversial. Is it because Lorentz forces cause a suppression of Lagrangian chaos (Cattaneo, Hughes & Kim 1996) or is it due to a non-linear addition to α due to helical parts of the small-scale magnetic field, as is argued here? The latter alternative is also supported by the excellent agreement between model calculations and simulations (Blackman & Brandenburg 2002; Field & Blackman 2002). Furthermore, the simulations of Brandenburg & Subramanian (2005b) demonstrate that this quenching is accompanied by an increase of $-\alpha_{\rm M}$ towards $\alpha_{\rm K}$, and that this $\alpha_{\rm K}$ itself is unquenched. Subsequent analysis of their data shows that $\alpha_{\rm K}$ remains unquenched regardless of whether or not one uses the proper anisotropic expression (Brandenburg & Subramanian, unpublished).

Our present results are of course restricted to the case of small magnetic and fluid Reynolds numbers. This means that we have not tested any of the actual closure assumptions, like the τ-approximation. Such tests have so far only been done numerically (Brandenburg, Käpylä & Mohammed 2004; Brandenburg & Subramanian 2005a). Clearly, at large magnetic and fluid Reynolds numbers the τ-approximation can no longer yield exact results. Nevertheless, it provides a very practical tool to estimate the mean field transport coefficients in a way that captures correctly some of the effects that enter in the case of large magnetic and fluid Reynolds numbers. In that respect, it has been quite successful in reproducing the catastrophic quenching result for periodic domains, as well as suggesting ways to alleviate such quenching.

In Section 2 we outlined the conditions under which double FOSA is valid as being basically the requirement that $R_{\rm m} \ll 1$ and $Re \ll 1$. A subtle point concerns the validity of retaining the linear Lorentz force term, $\boldsymbol{B}_0 \cdot \nabla \boldsymbol{b}$, while neglecting the non-linear advection, $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$, even though non-linear advection is small compared to the viscous dissipation for $Re \ll 1$. This assumption is valid provided $B_0\, b/l > u^2/l$, or, using $b \sim R_{\rm m}\, B_0$, $B_0^2\, R_{\rm m} > u^2$. (We thank Eric Blackman for pointing this out to us.) Note that in terms of the critical field $B_{\rm cr} = \sqrt{\nu\eta} q_0$, which divides the regimes where the Lorentz force is important ($B_0 > B_{\rm cr}$) and where it is not ($B_0 < B_{\rm cr}$), this requirement becomes $B_0/B_{\rm cr} > Re^{1/2}$. Therefore, for small fluid Reynolds number our assumption of retaining the linear Lorentz force term while dropping non-linear advection is indeed valid for most regimes of interest for α suppression. For smaller mean fields, where $B_0/B_{\rm cr} < Re^{1/2}$, in any case the Lorentz force has no impact. The interesting point seems to be that for low Reynolds number systems, the typical reference mean field is not the equipartition field $B_0 = u$, but rather $B_0 = B_{\rm cr} \sim u/(R_{\rm m}\, Re)^{1/2} > u$.

Throughout this work we have adopted an externally imposed body force to drive the flow. This is commonly done in many simulations in order to achieve homogeneous isotropic conditions that are amenable to analytic treatment. Clearly, this is not the case for many astrophysical flows that are driven by convection (e.g. in stars) or the magnetorotational instability (e.g. in accretion discs). Such flows tend to show long-range spatial correlations, which means that the alpha tensor should really be treated as a integral kernel (see e.g. Brandenburg & Sokoloff 2002). It is at present unclear whether such more natural forcings are closer to steady or to random forcing, and how big is the resulting $(\boldsymbol{f}, \boldsymbol{b})$ correlation. Given that this correlation represents already a linear effect, it is likely that the $\alpha_{\rm F}$ term can simply be subsumed into an expression for a modified kinetic $\alpha_{\rm K}$. If this is the case, we can continue to write $\alpha \approx \alpha_{\rm K} + \alpha_{\rm M}$ as the sum of a mildly suppressed kinetic part depending on the velocity field and a magnetic part, so that their sum accounts for the tendency towards catastrophic α-quenching in the absence of helicity fluxes. We recall that such a split is exact in the limit of delta-correlated forcing.

6 CONCLUSIONS

Our work was motivated in part by the detailed criticism expressed by Rädler & Rheinhardt (2007). In view of our new results we can now make the following statements for small magnetic and fluid Reynolds numbers. First, it is true that the $\hat{\alpha}_{\rm K}$ that is calculated under FOSA does indeed capture the full non-linear α-effect – provided it is based on the actual velocity field. Secondly, the $\alpha_{\rm K}$ that is calculated in the τ approximation, is not simply related to $\hat{\alpha}_{\rm K}$, except in the limit of steady forcing. Thirdly, in the limit of small magnetic and fluid Reynolds numbers, both FOSA and the τ-approximation give identical results. Indeed, all methods of calculating the α-effect agree, as they should, given that the starting equations were the same. However, the force–field correlation cannot be ignored in general. The exception is when one analyses the case of delta-correlated forcing in a manner akin to the τ-approximation, where the force–field correlation does vanish and hence $\alpha_{\rm F} = 0$ explicitly. In this case one can indeed write $\alpha = \alpha_{\rm K} + \alpha_{\rm M}$, or the sum of a kinetic and magnetic α-effects. Furthermore, due to the spatial non-locality of the Greens function for small magnetic and fluid Reynolds numbers, the τ-approximation should be carried out at the level of spectral correlation tensors, as is done here. Somewhat surprisingly, the delta-correlated forcing case yields an asymptotic $\alpha \propto B_0^{-2}$ scaling as opposed to the well-known $\alpha \propto B_0^{-3}$ behaviour for steady forcing.

Although our work is limited to small magnetic and fluid Reynolds numbers, the calculations of Method C and Method D, and its agreement with the results of Method A, (for both steady and time-dependent forcings), do suggest one way of thinking about the effect of Lorentz forces: they lead to a decrease of α predominantly by addition of terms proportional to the helical parts of the small-scale magnetic field. Hence getting rid of such small-scale magnetic helicity by corresponding helicity fluxes, may indeed be the way astrophysical dynamos avoid catastrophic quenching of α to make their dynamos work efficiently.

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