α-effect dynamos with zero kinetic helicity

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A simple explicit example of a Roberts-type dynamo is given in which the α effect of mean-field electrodynamics exists in spite of pointwise vanishing kinetic helicity of the fluid flow. In this way, it is shown that α-effect dynamos do not necessarily require nonzero kinetic helicity. A mean-field theory of Roberts-type dynamos is established within the framework of the second-order correlation approximation. In addition, numerical solutions of the original dynamo equations are given that are independent of any approximation of that kind. Both theory and numerical results demonstrate the possibility of dynamo action in the absence of kinetic helicity.

I. INTRODUCTION

The essential breakthrough in the understanding of the origin of the large-scale magnetic fields of cosmic objects came with the development of mean-field dynamo theory. A central component of this theory is the α effect, that is, a mean electromotive force with a component parallel or antiparallel to the mean magnetic field in a turbulently moving electrically conducting fluid. The α effect, which occurs naturally in inhomogeneous turbulence on a rotating body, is a crucial element of the dynamo mechanisms proposed and widely accepted for the Sun and planets, for other stellar objects and even for galaxies; see, e.g., [1–3].

In the majority of investigations, this effect has been merely calculated in the so-called second-order correlation approximation, or first-order smoothing approximation, which ignore all contributions of higher than second order in the turbulent part of the fluid velocity. Moreover in many cases attention has been focused on the high-conductivity limit only, which can be roughly characterized by short correlation times of the turbulent motion in comparison to the magnetic-field decay time for a turbulent eddy. Under these circumstances the α effect is closely connected with some average over the kinetic helicity of the turbulent motion, that is, over $\boldsymbol{u} \cdot \boldsymbol{\omega}$, where $\boldsymbol{u}$ means the turbulent part of the fluid velocity and $\boldsymbol{\omega}$ the corresponding vorticity, $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$; see, e.g., [1–5]. More precisely, the coefficient α for isotropic turbulence turns out to be equal to $-\frac{1}{3} \int_0^\infty \overline{\boldsymbol{u}(\boldsymbol{x},t) \cdot \boldsymbol{\omega}(\boldsymbol{x},t-\tau)} d\tau$, often expressed in the form $-\frac{1}{3} \overline{\boldsymbol{u} \cdot \boldsymbol{\omega}} \tau_c$ with equal arguments of $\boldsymbol{u}$ and $\boldsymbol{\omega}$ and some appropriate time τ_c. Here and in what follows overbars indicate averages. For anisotropic turbulence, the trace of the α tensor is just equal to three times this value of α.

These findings have been sometimes overinterpreted in the sense that mean-field dynamos, or even dynamos at all, might not work without kinetic helicity of the fluid motion, that is, if $\boldsymbol{u} \cdot \boldsymbol{\omega}$ vanishes. There exist, however, several counterexamples.

First, the mean-field dynamo theory offers also dynamo mechanisms without the α effect, for which the average of

the kinetic helicity may well be equal to zero. For instance, a combination of the so-called $\boldsymbol{\Omega} \times \boldsymbol{J}$ effect, which may occur even in the case of homogeneous turbulence in a rotating body, with shear associated with differential rotation can act as a dynamo [6–14]. The possibility of a dynamo due to turbulence influenced by large-scale shear and the shear flow itself [15–17] is still under debate [18–21].

Second, in dynamo theory beyond the mean-field concept a number of examples of dynamos without kinetic helicity are known. The dynamo proposed by Herzenberg [22], usually considered as the first existence proof for homogeneous dynamos at all, works without kinetic helicity. Likewise, in the dynamo models of Gailitis working with an axisymmetric meridional circulation in cylindrical or spherical geometry [23–25], there is no kinetic helicity. We mention here further the dynamo in a layer with hexagonal convection cells proposed by Zheligovsky and Galloway [26], in which the kinetic helicity is zero everywhere.

Third, it is known since the studies by Kazantsev [27] that small-scale dynamos may work without kinetic helicity. This fact has meanwhile been confirmed by many other investigations; see, e.g., [3].

Let us return to the mean-field concept but restrict ourselves, for the sake of simplicity, to homogeneous and statistically steady turbulence. We stay first with the second-order correlation approximation but relax the restriction to the high-conductivity limit. As is known from early studies, e.g., [1,12,28], the crucial parameter for the α effect is then no longer the mean kinetic helicity $\overline{\boldsymbol{u} \cdot \boldsymbol{\omega}}$. The α effect is in general determined by the function $h(\boldsymbol{\xi}, \tau) = \overline{\boldsymbol{u}(\boldsymbol{x},t) \cdot \boldsymbol{\omega}(\boldsymbol{x} + \boldsymbol{\xi}, t + \tau)}$ or, which is equivalent, by the kinetic helicity spectrum, defined as its Fourier transform with respect to $\boldsymbol{\xi}$ and τ. In the high-conductivity limit this brings us back to the above-mentioned results. In the low-conductivity limit, that is, large correlation times in comparison to the magnetic-field decay time for a turbulent eddy, the α effect is closely connected with the quantity $\overline{\boldsymbol{u} \cdot \boldsymbol{\psi}}$, where $\boldsymbol{\psi}$ is the vector potential of $\boldsymbol{u}$, that is $\nabla \times \boldsymbol{\psi} = \boldsymbol{u}$ with $\nabla \cdot \boldsymbol{\psi} = 0$. For isotropic turbulence the coefficient α is then equal to $-(1/3\eta) \overline{\boldsymbol{u} \cdot \boldsymbol{\psi}}$, where η means the magnetic diffusivity of the

fluid [1,4,5,28,29]. In the anisotropic case the trace of the α tensor is again equal to three times this value of α. In general, $\boldsymbol{u} \cdot \boldsymbol{\omega}$ does not vanish in this limit but it is without interest for the α effect. Both $\overline{\boldsymbol{u} \cdot \boldsymbol{\omega}}$ and $\overline{\boldsymbol{u} \cdot \boldsymbol{\psi}}$ are quantities that, if nonzero, indicate deviations of the turbulence from reflectional symmetry. In the second-order correlation approximation the $\boldsymbol{\Omega} \times \boldsymbol{J}$ effect, too, is completely determined by the kinetic helicity spectrum [12].

Beyond the second-order correlation approximation the α effect is no longer determined by the kinetic helicity spectrum alone. In an example given by Gilbert *et al.* [30] an α effect (with zero trace of the α tensor) occurs in a nonsteady flow even with zero kinetic helicity spectrum. Note that small-scale dynamos may also work with zero kinetic helicity spectrum, but they do not produce large-scale fields.

Interesting simple dynamo models have been proposed by Roberts [31,32]. In his second paper [32] he considered dynamos due to steady fluid flows which are periodic in two Cartesian coordinates, say x and y, but independent of the third one, z. When speaking of a "Roberts dynamo" in what follows we refer always to the first flow pattern envisaged there [Eq. (5.1), Fig. 1]. This dynamo played a central role in designing the homogeneous two-scale dynamo experiment of Müller and Stieglitz [33–36]. It can be easily interpreted within the mean-field concept. In that sense it occurs as an α-effect dynamo with an anisotropic α effect. The flow pattern proposed by Roberts, and in a sense realized in the experiment of Müller and Stieglitz, shows nonzero kinetic helicity. In this paper we want to demonstrate that this kind of dynamo works also with a slightly modified flow pattern in which the kinetic helicity is exactly equal to zero everywhere (but the kinetic helicity spectrum remains nonzero).

In Sec. II we present a simple mean-field theory of Roberts dynamos at the level of the second-order correlation approximation. In Sec. III we report on numerical results that apply independently of this approximation. Finally, in Sec. IV some conclusions are discussed.

II. A MODIFIED ROBERTS DYNAMO

A. Starting point

We focus our attention now on the Roberts dynamo in the above sense and refer again to a Cartesian coordinate system (x, y, z). The fluid is considered as incompressible. Therefore its velocity $\boldsymbol{u}$ has to satisfy $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$, and it is assumed to be a sum of two parts, one proportional to $\boldsymbol{e} \times \boldsymbol{\nabla} \chi$ and the other to $\chi \boldsymbol{e}$, where $\boldsymbol{e}$ is the unit vector in the z direction, and $\chi = \sin(\pi x/a) \sin(\pi y/a)$. The corresponding flow pattern is depicted in Fig. 1. We denote the sections defined by $na \leq x \leq (n+1)a$ and $ma \leq x \leq (m+1)a$ with integer n and m as "cells." In that sense the velocity $\boldsymbol{u}$ changes sign when we proceed from one cell to an adjacent one. A fluid flow of that kind indeed acts as a dynamo. Magnetic-field modes with the same periodicity in x and y but varying with z as $\sin(kz)$ or $\cos(kz)$ may grow for sufficiently small values of $|k|$. The most easily excitable modes for a given k possess a part which is independent of x and y, but varies with z.

B. Mean-field theory with a generalized fluid flow

Let us proceed to more general flow patterns which show some of the crucial properties of the specific flow pattern
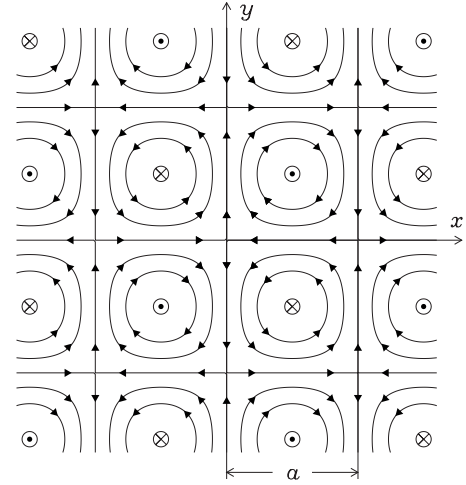

FIG. 1. The Roberts flow pattern.

considered so far. We assume again that the flow is incompressible, steady, depends periodically on x and y but is independent of z, and shows a cell structure as in the example above. More precisely, we require that $\boldsymbol{u}$ changes sign if the pattern is shifted by a length a along the x or y axis or is rotated by 90° about the z axis.

Let us sketch the mean-field theory of dynamos working with fluid flows of this type (partially using ideas described in our paper [37], in the following referred to as RB03).

It is assumed that the magnetic field $\boldsymbol{B}$ is governed by the induction equation

$$\eta \nabla^2 \boldsymbol{B} + \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) - \partial_t \boldsymbol{B} = \boldsymbol{0}, \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \quad (1)$$

with the magnetic diffusivity η being constant.

We define mean fields by averaging the original fields over a square that corresponds to four cells in the xy plane. More precisely, we define the mean field $\overline{F}$, which belongs to an original field F, in any given point $\boldsymbol{x} = \boldsymbol{x}_0$ by averaging F over the square defined by $x_0 - a \leq x \leq x_0 + a$ and $y_0 - a \leq y \leq y_0 + a$. If applied to quantities that are periodic in x and y with a period length $2a$, this average satisfies the Reynolds rules. Clearly we have $\overline{\boldsymbol{u}} = \boldsymbol{0}$.

When taking the average of Eq. (1) and denoting $\boldsymbol{B} - \overline{\boldsymbol{B}}$ by $\boldsymbol{b}$, we obtain the mean-field induction equation

$$\eta \nabla^2 \overline{\boldsymbol{B}} + \boldsymbol{\nabla} \times \boldsymbol{\mathcal{E}} - \partial_t \overline{\boldsymbol{B}} = \boldsymbol{0}, \quad \boldsymbol{\nabla} \cdot \overline{\boldsymbol{B}} = 0, \quad (2)$$

with the mean electromotive force

$$\boldsymbol{\mathcal{E}} = \overline{\boldsymbol{u} \times \boldsymbol{b}}. \quad (3)$$

For the determination of $\boldsymbol{\mathcal{E}}$ for a given $\boldsymbol{u}$ the equation for $\boldsymbol{b}$ is of interest. This equation follows from (1) and (2),

$$\eta \nabla^2 \boldsymbol{b} + \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b})' - \partial_t \boldsymbol{b} = -\boldsymbol{\nabla} \times (\boldsymbol{u} \times \overline{\boldsymbol{B}}),$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{b} = 0, \quad (4)$$

where $(\boldsymbol{u} \times \boldsymbol{b})'$ stands for $\boldsymbol{u} \times \boldsymbol{b} - \overline{\boldsymbol{u} \times \boldsymbol{b}}$.

For the sake of simplicity we consider here only the steady case. We introduce a Fourier transformation with respect to z, that is, we represent any function $F(x, y, z)$ by

$F(x,y,z)=\int\hat{F}(x,y,k)\exp(ikz)dk$. If $F(x,y,z)$ is real, we have $\hat{F}^*(x,y,k)=\hat{F}(x,y,-k)$, where the asterisk means complex conjugation. We have now

$$\hat{\mathcal{E}}=\overline{\boldsymbol{u}\times\hat{\boldsymbol{b}}}. \tag{5}$$

Transforming (4) accordingly, we obtain

$$\eta(\nabla^2-k^2)\hat{\boldsymbol{b}}+(\boldsymbol{\nabla}+ik\boldsymbol{e})\times(\boldsymbol{u}\times\hat{\boldsymbol{b}})'=-(\boldsymbol{\nabla}+ik\boldsymbol{e})\times(\boldsymbol{u}\times\hat{\overline{\boldsymbol{B}}}),$$

$$\boldsymbol{\nabla}\cdot\hat{\boldsymbol{b}}+ik\hat{b}_z=0. \tag{6}$$

We further suppose that $\hat{\overline{\boldsymbol{B}}}$ is independent of x and y. This implies that $\hat{\mathcal{E}}$ is independent of x and y, too. It is then obvious that the connection between $\hat{\mathcal{E}}$ and $\hat{\overline{\boldsymbol{B}}}$ must have the form

$$\hat{\mathcal{E}}_i=\hat{a}_{ij}\hat{\overline{B}}_j, \tag{7}$$

where $\hat{a}_{ij}$ is, like $\hat{\mathcal{E}}$ and $\hat{\overline{\boldsymbol{B}}}$, independent of x and y, but depends on k. Because $\hat{\mathcal{E}}_i^*(k)=\hat{\mathcal{E}}_i(-k)$ and $\hat{\overline{B}}_i^*(k)=\hat{\overline{B}}_i(-k)$ it also has to satisfy $\hat{a}_{ij}^*(k)=\hat{a}_{ij}(-k)$.

Clearly $\hat{a}_{ij}$ is determined by $\boldsymbol{u}$. Since $\mathcal{E}$ does not change on inverting the sign of $\boldsymbol{u}$, and a 90° rotation of the $\boldsymbol{u}$ field about the z axis changes nothing other than its sign, the components of the tensor $\hat{a}_{ij}$ have to be invariant under 90° rotations of the coordinate system about the z axis. We may therefore conclude that

$$\hat{a}_{ij}=a_1(|k|)\delta_{ij}+a_2(|k|)e_ie_j+ia_3(|k|)k\epsilon_{ijk}e_k \tag{8}$$

with real a_1, a_2, and a_3. Together with (7) this leads to $\hat{\mathcal{E}}_z=(a_1+a_2)\hat{\overline{B}}_z$. On the other hand, $\hat{\mathcal{E}}_z$ is equal to the average of $u_x\hat{b}_y-u_y\hat{b}_x$, and we may conclude from (6) that $\hat{b}_x$ and $\hat{b}_y$ are independent of $\hat{\overline{B}}_z$. This implies $a_1+a_2=0$. We may then write

$$\hat{a}_{ij}=-\hat{\alpha}_\perp(k)(\delta_{ij}-e_ie_j)+i\hat{\beta}(k)k\epsilon_{ijk}e_k \tag{9}$$

with two real quantities $\hat{\alpha}_\perp$ and $\hat{\beta}$, which are even functions of k. From (7) and (9) we obtain

$$\hat{\mathcal{E}}=-\hat{\alpha}_\perp(k)[\hat{\overline{\boldsymbol{B}}}-(\boldsymbol{e}\cdot\hat{\overline{\boldsymbol{B}}})\boldsymbol{e}]-i\hat{\beta}(k)k\boldsymbol{e}\times\hat{\overline{\boldsymbol{B}}}. \tag{10}$$

This relation is discussed in some detail in RB03, Sec. V B.

We now restrict our attention to the limit of small k, that is, small variations of $\overline{\boldsymbol{B}}$ with z. Then $\hat{\alpha}_\perp$ and $\hat{\beta}$ lose their dependence on k. Denoting them by $\alpha_\perp$ and β, we conclude from (10) that

$$\mathcal{E}=-\alpha_\perp[\overline{\boldsymbol{B}}-(\boldsymbol{e}\cdot\overline{\boldsymbol{B}})\boldsymbol{e}]-\beta\boldsymbol{e}\times d\overline{\boldsymbol{B}}/dz. \tag{11}$$

Equation (2) together with (11) allow solutions of the form $\overline{\boldsymbol{B}}=\mathrm{Re}[\overline{\boldsymbol{B}}_0\exp(ikz+\lambda t)]$ with $\lambda=\pm\alpha_\perp k-(\eta+\beta)k^2$. Growing solutions are possible if

$$\frac{|\alpha_\perp|}{|k|(\eta+\beta)}>1. \tag{12}$$

We note that, in agreement with the antidynamo theorem by Zeldovich, λ vanishes if $k\to0$, that is, if $\overline{\boldsymbol{B}}$ loses its dependence on z.

C. Calculation of $\alpha_\perp$ and β

For the determination of the coefficients $\alpha_\perp$ and β we restrict ourselves to an approximation which corresponds to the second-order correlation approximation. It is defined by the neglect of $(\boldsymbol{u}\times\boldsymbol{b})'$ in (4), or $(\boldsymbol{u}\times\hat{\boldsymbol{b}})'$ in (6). In view of $\alpha_\perp$ and β, which correspond to the limit of small k, we expand $\hat{\boldsymbol{b}}$ and $\hat{\overline{\boldsymbol{B}}}$ in powers of k but neglect all contributions with higher than first powers of k. That is, $\hat{\boldsymbol{b}}=\hat{\boldsymbol{b}}^{(0)}+k\hat{\boldsymbol{b}}^{(1)}$ and $\hat{\overline{\boldsymbol{B}}}=\hat{\overline{\boldsymbol{B}}}^{(0)}+k\hat{\overline{\boldsymbol{B}}}^{(1)}$, where of course $\hat{\boldsymbol{b}}^{(0)}=\boldsymbol{b}^{(0)}$ and $\hat{\overline{\boldsymbol{B}}}^{(0)}=\overline{\boldsymbol{B}}^{(0)}$. From (6) it follows then

$$\eta\nabla^2\hat{\boldsymbol{b}}^{(0)}=-(\hat{\overline{\boldsymbol{B}}}^{(0)}\cdot\boldsymbol{\nabla})\boldsymbol{u},$$

$$\eta\nabla^2\hat{\boldsymbol{b}}^{(1)}=-i\boldsymbol{e}\times(\boldsymbol{u}\times\hat{\overline{\boldsymbol{B}}}^{(0)})-(\hat{\overline{\boldsymbol{B}}}^{(1)}\cdot\boldsymbol{\nabla})\boldsymbol{u}. \tag{13}$$

We put now $\boldsymbol{u}=\boldsymbol{\nabla}\times\boldsymbol{\psi}$, where $\boldsymbol{\psi}$ denotes a vector potential satisfying $\boldsymbol{\nabla}\cdot\boldsymbol{\psi}=0$; further $\boldsymbol{\psi}=\boldsymbol{\nabla}\times\tilde{\boldsymbol{\psi}}$ with $\boldsymbol{\nabla}\cdot\tilde{\boldsymbol{\psi}}=0$, which implies

$$\boldsymbol{u}=-\nabla^2\tilde{\boldsymbol{\psi}}. \tag{14}$$

From (13) with $\boldsymbol{u}$ expressed in this way we conclude

$$\hat{\boldsymbol{b}}^{(0)}=\frac{1}{\eta}(\hat{\overline{\boldsymbol{B}}}^{(0)}\cdot\boldsymbol{\nabla})\tilde{\boldsymbol{\psi}},$$

$$\hat{\boldsymbol{b}}^{(1)}=\frac{1}{\eta}[i\boldsymbol{e}\times(\tilde{\boldsymbol{\psi}}\times\hat{\overline{\boldsymbol{B}}}^{(0)})-(\hat{\overline{\boldsymbol{B}}}^{(1)}\cdot\boldsymbol{\nabla})\tilde{\boldsymbol{\psi}}]. \tag{15}$$

Putting in the above sense also $\hat{\mathcal{E}}=\hat{\mathcal{E}}^{(0)}+k\hat{\mathcal{E}}^{(1)}$, we find

$$\hat{\mathcal{E}}^{(0)}=\frac{1}{\eta}\overline{\boldsymbol{u}\times(\hat{\overline{\boldsymbol{B}}}^{(0)}\cdot\boldsymbol{\nabla})\tilde{\boldsymbol{\psi}}},$$

$$\hat{\mathcal{E}}^{(1)}=-\frac{1}{\eta}\{\overline{i\boldsymbol{e}\times[\boldsymbol{u}\times(\tilde{\boldsymbol{\psi}}\times\hat{\overline{\boldsymbol{B}}}^{(0)})]}+\overline{\boldsymbol{u}\times(\hat{\overline{\boldsymbol{B}}}^{(1)}\cdot\boldsymbol{\nabla})\tilde{\boldsymbol{\psi}}}\}. \tag{16}$$

We compare this with what follows from (10) in this expansion,

$$\hat{\mathcal{E}}^{(0)}=-\alpha_\perp[\hat{\overline{\boldsymbol{B}}}^{(0)}-(\boldsymbol{e}\cdot\hat{\overline{\boldsymbol{B}}}^{(0)})\boldsymbol{e}],$$

$$\hat{\mathcal{E}}^{(1)}=-i\beta\boldsymbol{e}\times\hat{\overline{\boldsymbol{B}}}^{(0)}-\alpha_\perp[\hat{\overline{\boldsymbol{B}}}^{(1)}-(\boldsymbol{e}\cdot\hat{\overline{\boldsymbol{B}}}^{(1)})\boldsymbol{e}]. \tag{17}$$

In this way we obtain first

$$\alpha_\perp=-\frac{1}{\eta}\overline{[\boldsymbol{u}\times(\boldsymbol{g}\cdot\boldsymbol{\nabla})\tilde{\boldsymbol{\psi}}]}\cdot\boldsymbol{g},$$
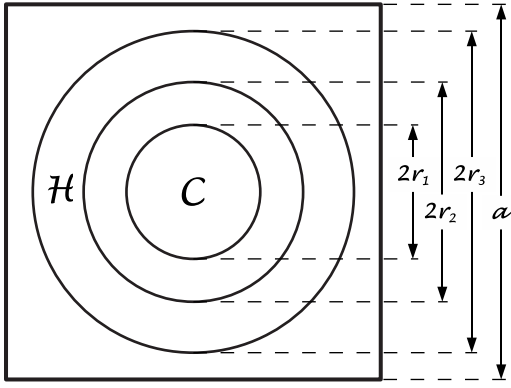
FIG. 2. A single cell of the modified Roberts flow.

$$\beta = -\frac{1}{\eta}\overline{[u \cdot \widetilde{\psi} - \overline{(h \cdot u)(h \cdot \widetilde{\psi})}]}, \tag{18}$$

where g and h are unit vectors in the directions of $\hat{\bar{B}}^{(0)}$ $-(e \cdot \hat{\bar{B}}^{(0)})e$ and $e \times \hat{\bar{B}}^{(0)}$, respectively. Of course, $\alpha_\perp$ and β cannot really depend on these directions, and we may average the two expressions for $\alpha_\perp$ with $g=(1,0,0)$ and $g=(0,1,0)$ and likewise those for β with $h=(1,0,0)$ and $h=(0,1,0)$. This yields

$$\alpha_\perp = \frac{1}{2\eta}\overline{u \cdot (\nabla \times \widetilde{\psi})},$$

$$\beta = -\frac{1}{2\eta}\overline{[u \cdot \widetilde{\psi} + \overline{(e \cdot u)(e \cdot \widetilde{\psi})}]}. \tag{19}$$

The result for $\alpha_\perp$ can also be written in the form

$$\alpha_\perp = \frac{1}{2\eta}\overline{u \cdot \psi} = \frac{1}{2\eta}\overline{\psi \cdot (\nabla \times \psi)}. \tag{20}$$

In agreement with our remarks in the Introduction, the α effect is not determined by the average of the kinetic helicity $\overline{u \cdot \omega}$ but by the related but different quantity $\overline{u \cdot \psi}$, that is, some kind of mean helicity of the vector potential ψ.

D. A specific flow

We elaborate now on the above result for $\alpha_\perp$ for flow patterns similar to those that have been considered in the context of the aforementioned homogeneous two-scale dynamo experiment [38]. To define the flow first in a single cell like those depicted in Fig. 2 we introduce a cylindrical coordinate system (r, φ, z) with $r=0$ in the middle of this cell and z as in the Cartesian system used above. We put then

$$u_r = 0 \quad \text{everywhere,}$$

$$u_\varphi = 0, \quad u_z = u \quad \text{in } r \leq r_1,$$

$$u_\varphi = 0, \quad u_z = 0 \quad \text{in } r_1 < r < r_2,$$

$$u_\varphi = \omega r, \quad u_z = \varepsilon \omega a/2 \quad \text{in } r_2 \leq r \leq r_3, \tag{21}$$

with ω and ε being constants. The full flow pattern is defined by continuation of that in the considered cell to the other cells such that the flow always has different signs in two adjacent cells. Thinking of the "central channels" and the "helical channels" of the device of Müller and Stieglitz, we label the flow regions $r \leq r_1$ and $r_2 \leq r \leq r_3$ by $\mathcal{C}$ and $\mathcal{H}$, respectively. Clearly, finite kinetic helicity exists only in $\mathcal{H}$, and only for nonzero ε. Note that, in contrast to the situation depicted in Fig. 1, here the kinetic helicity is non-negative as long as ε is positive.

For the flow defined in this way quantities like $\overline{u \cdot \omega}$ or $\overline{u \cdot \psi}$ depend on u and ω. In the following, we express u by the flow rate V_C of fluid through the cross section $0 \leq r \leq r_1$ at a given z, and ω by the flow rate V_H of fluid through a meridional surface defined by $r_2 \leq r \leq r_3$ and $0 \leq z \leq a$ and a given φ. As can be easily verified we have then

$$\overline{u \cdot \omega} = c\varepsilon V_H^2, \tag{22}$$

with some positive quantity c depending on r_2, r_3, and a. For $\overline{u \cdot \psi}$ it is important that the part of the vector potential ψ resulting from the flow in the region $\mathcal{C}$ of a given cell does not vanish in $\mathcal{H}$, and likewise the part resulting from the flow in $\mathcal{H}$ does not vanish in $\mathcal{C}$. Moreover, the parts of ψ resulting from flows in other cells have to be taken into account. Considering these aspects, we may conclude that

$$\overline{u \cdot \psi} = (c_1 V_C + \varepsilon c_2 V_H)V_H \tag{23}$$

with nonzero quantities c_1 and c_2 depending on r_1, r_2, r_3, and a.

Clearly, $\overline{u \cdot \psi}$ and, according to (20), $\alpha_\perp$ remain in general different from zero if the flow loses its kinetic helicity, that is, as $\overline{u \cdot \omega}$, or ε, vanishes. This implies the possibility of α-effect dynamos without kinetic helicity.

III. NUMERICAL EXAMPLES OF α-EFFECT DYNAMOS WITH AND WITHOUT KINETIC HELICITY

In our above calculations of $\alpha_\perp$ and β an approximation in the spirit of the second-order correlation approximation has been used. The results are reliable only if a suitably defined magnetic Reynolds number is much smaller than unity. This can also be expressed by requiring that the normalized flow rates $\widetilde{V}_C = V_C/\eta a$ and $\widetilde{V}_H = V_H/\eta a$ show this property. In addition we have assumed weak variations of the mean magnetic field in the z direction; more precisely, $\kappa = k/a \ll 1$.

In order to confirm the existence of α-effect dynamos without kinetic helicity in an independent way and to check whether it occurs only under the mentioned conditions or over a wider range of parameters, Eq. (1) with the flow defined by (21) has been solved numerically. The same numerical method as in RB03 has been used.

In Table I and Fig. 3 some numerically determined dimensionless flow rates $\widetilde{V}_C$ and $\widetilde{V}_H$ are given for marginal dynamo states. For case (i) a nonzero kinetic helicity was chosen, for the cases (ii) and (iii) zero kinetic helicity. The marginal modes turned out to be nonoscillatory. Note that the parameters for case (i), $2r_1/a = 2r_2/a = 0.5$, $2r_3/a = 1$, $\varepsilon = 0.228$, and

TABLE I. Examples of marginal dynamo states with $\kappa = 0.9$.

Case	$2r_1/a$	$2r_2/a$	$2r_3/a$	ε	$\widetilde{V}_C$	$\widetilde{V}_H$
(i)	0.5	0.5	1.0	0.228	2	0.736
(ii)	0.5	0.5	1.0	0	2	0.805
(iii)	0.5	0.7	1.0	0	2	0.965

$\kappa = 0.9$, are realistic values in view of the dynamo device of Müller and Stieglitz [38]. In cases (i) and (ii) the regions $\mathcal{C}$ and $\mathcal{H}$ have a common surface. In order to make sure that there is no effect of an artificial kinetic helicity due to the restricted numerical resolution at this surface, in case (iii) a clear separation of these two regions was considered.

As long as the kinetic helicity in $\mathcal{H}$ does not vanish, that is, $\varepsilon \neq 0$, the dynamo works even with $V_C = 0$, that is, zero flow in $\mathcal{C}$. In the absence of kinetic helicity, $\varepsilon = 0$, the dynamo works, too, as long as V_C and therefore $\boldsymbol{u} \cdot \boldsymbol{\psi}$ are different from zero. However, the dynamo disappears if, in addition to ε, also V_C and so $\boldsymbol{u} \cdot \boldsymbol{\psi}$ vanish. This is plausible since then choices of the coordinate origin $\boldsymbol{x} = \boldsymbol{0}$ are possible such that $\boldsymbol{u}(\boldsymbol{x})$ and $-\boldsymbol{u}(-\boldsymbol{x})$ coincide. In this case, usually referred to as "parity invariant," any α effect can be excluded.

The dissipation of the mean magnetic field grows with κ. For this reason the dynamo threshold grows with κ, too.

IV. CONCLUSIONS

We have examined a modified version of the Roberts dynamo with a steady fluid flow as it was considered in the context of the dynamo experiment of Müller and Stieglitz. By contrast to what might be suggested by simple explanations of this dynamo, the necessary deviation of the flow from reflectional symmetry is not adequately described by the kinetic helicity $\boldsymbol{u} \cdot \boldsymbol{\omega}$ of the fluid flow or its average $\overline{\boldsymbol{u} \cdot \boldsymbol{\omega}}$. Dynamo action with steady fluid flow requires in truth a deviation which is indicated by a nonzero value of the quantity $\overline{\boldsymbol{u} \cdot \boldsymbol{\psi}}$, where $\boldsymbol{\psi}$ is the vector potential of $\boldsymbol{u}$. Even if the kinetic helicity is equal to zero everywhere, dynamo action occurs if only the quantity $\boldsymbol{u} \cdot \boldsymbol{\psi}$, the helicity of the vector potential of the flow, is unequal to zero. This fact is not surprising in light of the general findings of mean-field electrodynamics.
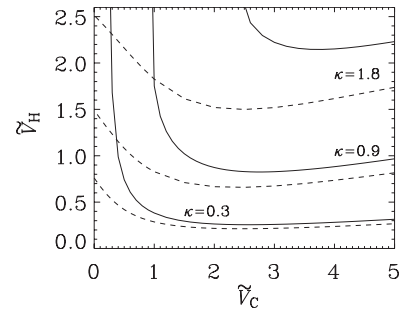

FIG. 3. Lines of marginal stability in the $\widetilde{V}_C$-$\widetilde{V}_H$ plane for the cases (i) (dashed lines) and (iii) (solid lines) defined in Table I, and three values of κ.

We stress that our result does not mean that the quantity $\overline{\boldsymbol{u} \cdot \boldsymbol{\psi}}$ is in general more fundamental for the α effect and its dynamo action than $\overline{\boldsymbol{u} \cdot \boldsymbol{\omega}}$. In cases with steady fluid flow indeed $\overline{\boldsymbol{u} \cdot \boldsymbol{\psi}}$ is crucial. However, with flows varying rapidly in time, the crucial quantity is $\overline{\boldsymbol{u} \cdot \boldsymbol{\omega}}$.

Both the mean kinetic helicity $\overline{\boldsymbol{u} \cdot \boldsymbol{\omega}}$ and the quantity $\overline{\boldsymbol{u} \cdot \boldsymbol{\psi}}$ are determined by the helicity spectrum of the flow. That is, the Roberts dynamo does not work with a zero kinetic helicity spectrum. In this sense the situation is different from that in the interesting example of an α-effect dynamo given by Gilbert et al. [30], which works with a very special non-steady flow for which the kinetic helicity spectrum is indeed zero.

Looking back at the experiment of Müller and Stieglitz we may state that a modified (technically perhaps more complex) version without kinetic helicity of the fluid flow should show the dynamo effect, too. As Table I exemplifies, however, the kinetic helicity can well reduce the dynamo threshold. In that sense we may modify the title of the paper of Gilbert et al. and say: Helicity is unnecessary for Roberts dynamos—but it helps.

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